Pontificia Universidad Católica de Chile
Facultad de Física - Instituto de Astrofísica
ASP5107 - Dynamics of stellar and planetary systems
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Dynamics of Stellar and Planetary systems - Solved problems

Problem 1: Molniya orbits

The quadrupolar deformation of Earth (J_2) gradually changes the argument of perigee ω of Earth Satellites. For some communication satellites this is energy inefficient as constant orbit corrections would be required with station-keeping thruster burns. But our Russian colleagues figured out a smart way of placing satellites into some peculiar *frozen orbits*: the Molniya orbits. Let us characterize them.

a) Consider the gravitational potential:

$$\phi(r) = -\frac{\mathcal{G}M_\oplus}{r}\left[1 - J_2\left(\frac{R_\oplus}{r}\right)^2 P_2(\cos I)\right] \tag{1}$$

where I is the Satellite's inclination relative the Earth's equator, $J_2 \simeq 10^{-3}$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$ is a Legendre polynomial. Write down the orbit-averaged Hamiltonian in terms of the Delaunay momenta: L, G, H.

We can rewrite the given potential as

$$\begin{aligned}\phi(r) &= -\frac{\mathcal{G}M_\oplus}{r} + \frac{\mathcal{G}M_\oplus J_2 R_\oplus^2(3\cos^2 I - 1)}{2r^3}\\ \mathcal{H} &= \mathcal{H}_K + \mathcal{H}_1\end{aligned}$$

Then the orbit-averaged Hamiltonian is just

$$\langle\mathcal{H}\rangle = \langle\mathcal{H}_1\rangle = \left\langle\frac{\mathcal{G}M_\oplus J_2 R_\oplus^2(3\cos^2 I - 1)}{2r^3}\right\rangle = \frac{\mathcal{G}M_\oplus J_2 R_\oplus^2}{2}\left(3\left\langle\frac{\cos^2 I}{r^3}\right\rangle - \left\langle\frac{1}{r^3}\right\rangle\right)$$

From the ISIMA Lectures we know that

$$\begin{aligned}\left\langle\frac{\cos^2 I}{r^3}\right\rangle &= \frac{\sin^2 I}{2a^3(1-e^2)^{3/2}}\\ \left\langle\frac{1}{r^3}\right\rangle &= \frac{1}{a^3(1-e^2)^{3/2}}\end{aligned}$$

Then replacing these expressions into our Hamiltonian we get

$$\langle \mathcal{H} \rangle = \frac{\mathcal{G} M_\oplus J_2 R_\oplus^2}{4a^3(1-e^2)^{3/2}} \left(3\sin^2 I - 2\right)$$

To write this in terms of the Delaunay Momenta, we remember that

$$L = \sqrt{\mathcal{G} M a} \qquad G = L\sqrt{1-e^2} \qquad H = G\cos I$$

Then

$$(1-e^2)^{3/2} = \left(\frac{G}{L}\right)^3$$
$$a^3 = \left(\frac{L^2}{\mathcal{G} M_\oplus}\right)^3$$
$$\sin^2 I = 1 - \cos^2 I = 1 - \left(\frac{H}{G}\right)^2$$

Hence we have

$$\langle \mathcal{H} \rangle = \frac{\mathcal{G} M_\oplus J_2 R_\oplus^2}{4\left(\frac{L^2}{\mathcal{G} M_\oplus}\right)^3 \left(\frac{G}{L}\right)^3} \left(3\left(1 - \left(\frac{H}{G}\right)^2\right) - 2\right)$$

$$= \frac{\left(\mathcal{G} M_\oplus\right)^4 J_2 R_\oplus^2}{4L^3 G^5} (G^2 - 3H^2)$$

b) Use Hamilton's equations to determine the critical inclination at which the Satellite does not precess: $\dot{\omega} = 0$.

We know that Hamilton's equations are:

$$\frac{dL}{dt} = -\frac{\partial H_{sec}}{\partial \ell}$$
$$\frac{dG}{dt} = -\frac{\partial H_{sec}}{\partial \omega}$$
$$\frac{dH}{dt} = -\frac{\partial H_{sec}}{\partial \Omega}$$

Then we know that

$$\dot{\omega} = \frac{d\omega}{dt} = \frac{\partial H_{sec}}{\partial G} = \frac{\left(\mathcal{G} M_\oplus\right)^4 J_2 R_\oplus^2}{4L^3} \frac{\partial}{\partial G} \frac{(G^2 - 3H^2)}{G^5}$$

Then

$$\dot{\omega} = \frac{\left(\mathcal{G} M_\oplus\right)^4 J_2 R_\oplus^2}{4L^3} \left(-\frac{3}{G^4} + 15\frac{H^2}{G^6}\right) = \frac{3\left(\mathcal{G} M_\oplus\right)^4 J_2 R_\oplus^2}{4L^3 G^4} \left(5\frac{H^2}{G^2} - 1\right)$$

To determine the critical inclination, we rewrite the expression as

$$\dot{\omega} = \frac{3\left(\mathcal{G} M_\oplus\right)^4 J_2 R_\oplus^2}{4(\mathcal{G} M_\oplus a)^{3/2}(\mathcal{G} M_\oplus a(1-e^2))^2} \left(5\cos^2 I - 1\right)$$
$$= \frac{3\left(\mathcal{G} M_\oplus\right)^{1/2} J_2 R_\oplus^2}{4a^{7/2}(1-e^2)^2} \left(5\cos^2 I - 1\right)$$

Then the critical inclination at which the Satellite does not precess is

$$I = \cos^{-1}\left(\sqrt{\frac{1}{5}}\right) = 63.4^\circ$$

c) The Molniya orbits have a high eccentricities (assume $e = 0.74$) to maximize the time-coverage of one Hemisphere, spending little time in the opposite hemisphere. Its sidereal period is 12 hours. Compute the rate of change of the longitude of the ascending node $\dot{\Omega}$ in degrees per day.

In this case, we are looking for

$$\dot{\Omega} = \frac{d\Omega}{dt} = \frac{\partial H_{sec}}{\partial H} = \frac{(\mathcal{G}M_\oplus)^4 J_2 R_\oplus^2}{4L^3} \frac{\partial}{\partial H} \frac{(G^2 - 3H^2)}{G^5}$$

Then

$$\dot{\Omega} = \frac{(\mathcal{G}M_\oplus)^4 J_2 R_\oplus^2}{4L^3} \frac{(-6H)}{G^5}$$

Which can be rewritten as

$$\begin{aligned}\dot{\Omega} &= -\frac{(\mathcal{G}M_\oplus)^4 J_2 R_\oplus^2}{4(\mathcal{G}M_\oplus a)^{3/2}} \frac{(6\cos I)}{(\mathcal{G}M_\oplus a(1-e^2))^2} \\ &= -\frac{3\,(\mathcal{G}M_\oplus)^{1/2} J_2 R_\oplus^2}{2a^{7/2}(1-e^2)^2}\cos I\end{aligned}$$

Considering

$$a = \left(\frac{P^2 \mathcal{G}M}{4\pi^2}\right)^{1/3}$$

We have

$$\dot{\Omega} = -\frac{3J_2 R_\oplus^2}{2(\mathcal{G}M)^{2/3}(1-e^2)^2}\left(\frac{2\pi}{P}\right)^{7/3}\cos I$$

Replacing with

$$\begin{aligned} e &= 0.74 \\ J_2 &= 10^{-3} \\ P &= 12h \\ \cos I &= \frac{1}{\sqrt{5}} \end{aligned}$$

We get $\dot{\Omega} \approx -0.136$ degrees per day

Problem 2: Orbits in the galactic center

The Galactic center hosts a supermassive black hole with mass of $M_\bullet = 4\times 10^6 M_\odot$ and a nuclear star cluster. We shall study the orbits in this environment assuming that the cluster can be modeled using a Hernquist potential:

$$\rho_H(r) = \frac{\rho_0}{(r/s)(1+r/s)^3} \tag{2}$$

a) Calculate the enclosed mass $M_H(r)$ and normalize ρ_H such that $M_H(\infty) = M_c$. Compute the potential $\phi_H(r)$.

In order to calculate the enclosed mass within an arbitrary radius, we integrate the density $\rho_H(r)$ from the origin (0) to the radius r, considering spherical coordinates. Then

$$\begin{aligned}
M_H(r) &= 4\pi \int_0^r r'^2 \rho_H(r') dr' \\
&= 4\pi \int_0^r \frac{r'^2\ \rho_0}{(r'/s)(1+r'/s)^3} dr' \\
&= 4\pi\rho_0 s \int_0^r \frac{r'}{(1+r'/s)^3} dr'
\end{aligned}$$

Changing variables to $u = 1 + r'/s$ so that $du = dr'/s$ we have:

$$\begin{aligned}
M_H(r) &= 4\pi\rho_0 s \int_1^{1+r/s} \frac{s(u-1)}{u^3}\ s\ du \\
&= 4\pi\rho_0 s^3 \left[\int_1^{1+r/s} \frac{1}{u^2}\ du - \int_1^{1+r/s} \frac{1}{u^3}\ du\right] \\
&= 4\pi\rho_0 s^3 \left[-u^{-1}\Big|_1^{1+r/s} - \frac{u^{-2}}{-2}\Big|_1^{1+r/s}\right] \\
&= 4\pi\rho_0 s^3 \left[\frac{-1}{(r/s+1)} + 1 - \left(\frac{1}{-2(r/s+1)^2} + \frac{1}{2}\right)\right] \\
&= 4\pi\rho_0 s^3 \left[\frac{-2(r/s+1) + 2(r/s+1)^2 + 1 - (r/s+1)^2}{2(r/s+1)^2}\right] \\
&= 4\pi\rho_0 s^3 \frac{r^2}{2s^2(r/s+1)^2}
\end{aligned}$$

Then

$$M_H(r) = \frac{2\pi\rho_0\ r^2\ s}{(r/s+1)^2}$$

To normalize ρ_H such that $M_H(\infty) = M_c$ we must take the limit:

$$
\begin{aligned}
M_c &= \lim_{r\to\infty} M_H \\
&= 2\pi\rho_0 s \lim_{r\to\infty} \frac{r^2}{(r/s+1)^2} \\
&= 2\pi\rho_0 s \lim_{r\to\infty} \frac{r^2}{r^2/s^2 + 2r/s + 1} \\
&= 2\pi\rho_0 s \lim_{r\to\infty} \frac{r^2}{r^2(1/s^2 + 2/rs + 1/r^2)} \\
&= 2\pi\rho_0 s \lim_{r\to\infty} \frac{1}{(1/s^2 + \cancelto{0}{2/rs} + \cancelto{0}{1/r^2})} \\
&= 2\pi\rho_0 s^3
\end{aligned}
$$

So

$$
\rho_0 = \frac{M_c}{2\pi s^3},
$$

$$
M_H = \frac{M_c\ (r/s)^2}{(r/s+1)^2}
$$

and

$$
\rho_H = \frac{M_c}{2\pi r s^2 (r/s+1)^3}
$$

Now we can compute the potential as:

$$
\begin{aligned}
\phi(r) &= -G \int_r^\infty \frac{M(r')dr'}{r'^2} \\
&= -G \int_r^\infty \frac{M_c\ (r'/s)^2}{r'^2(r'/s+1)^2} dr' \\
&= -\frac{G\ M_c}{s^2} \int_r^\infty \frac{1}{(r'/s+1)^2} dr'
\end{aligned}
$$

Changing variables as we did before we have:

$$
\begin{aligned}
\phi(r) &= -\frac{G\ M_c}{s^2} \int_{r/s+1}^\infty \frac{1}{u^2} s\ du \\
&= -\frac{G\ M_c}{s} \int_{r/s+1}^\infty \frac{du}{u^2} \\
&= -\frac{G\ M_c}{s} \left(-\frac{1}{u}\bigg|_{r/s+1}^\infty \right) \\
&= -\frac{G\ M_c}{s} \frac{1}{(r/s+1)} \\
&= -\frac{GM_c}{r+s}
\end{aligned}
$$

b) Calculate the circular velocity v_c, escape velocity v_e and circular orbital period T of the potential due to both $M_\bullet$ and the cluster. Plot their values for in the range of $r = 0.1 - 10$ pc in km/s results using $s = 4$ pc and $M_c = 8M_\bullet$.

We know that

$$v_c(r) = \sqrt{r\frac{d\phi}{dr}}$$

$$v_e(r) = \sqrt{2|\phi(r)|}$$

And

$$T(r) = \frac{2\pi r}{v_c}$$

The potential of our problem is the combination of the Keplerian potential due to the Black Hole and the Hernquist potential of the cluster. Then

$$\begin{aligned}\phi &= \phi_H + \phi_\bullet \\ &= \frac{-GM_c}{r+s} - \frac{GM_\bullet}{r}\end{aligned}$$

So

$$\frac{d\phi}{dr} = \frac{GM_c}{(r+s)^2} + \frac{GM_\bullet}{r^2}$$

Then

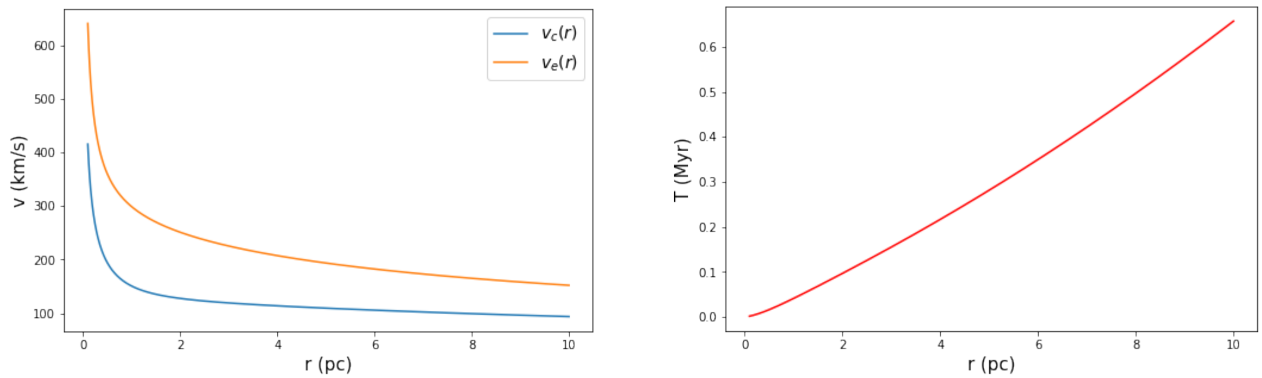
$$v_c = \sqrt{\frac{GM_c r^2 + GM_\bullet (s+r)^2}{r(s+r)^2}}$$

$$v_e = \sqrt{2\left(\frac{GM_c}{s+r} + \frac{GM_\bullet}{r}\right)}$$

And

$$\begin{aligned}T &= 2\pi r\sqrt{\frac{r(s+r)^2}{GM_c r^2 + GM_\bullet (s+r)^2}} \\ &= 2\pi\sqrt{\frac{r^3(s+r)^2}{GM_c r^2 + GM_\bullet (s+r)^2}}\end{aligned}$$

Plotting this values for r in the range $r = 0.1 - 10pc$ using $s = 4$ pc and $M_c = 8M_\bullet$ we have:

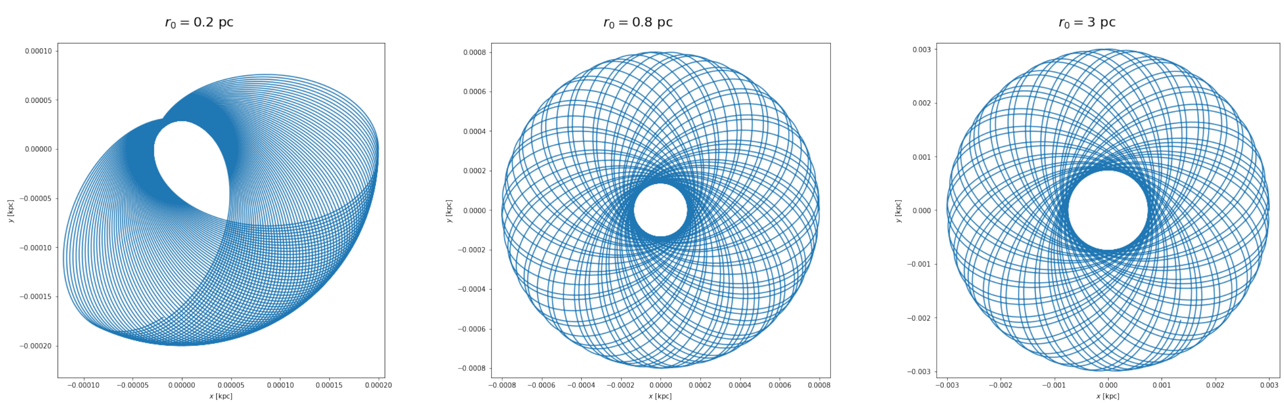


(a) Velocities **(b)** Circular orbital period

Figure 1: $v_c(r)$, $v_e(r)$ and $T(r)$ values for r in range 0.01 - 10 pc.

c) Use Gala **(or Galpy) to integrate the orbits at different locations:** $(0.2, 0, 0)$ **pc,** $(0.8, 0, 0)$ **pc, and** $(3, 0, 0)$ **pc up to** $t_{\max} = 30T$ **each. Set the velocity to** $(0, 0.5v_c, 0)$ **and the timestep to** $dt = 10^{-3}T$**. Discuss the behavior of the orbits as we move away from** $M_\bullet$**.**

Considering the three initial conditions given above and computing both v_c and T for each initial distance from the origin we get the following orbit plots:



(a) Initial distance $r_0 = 0.2$ pc **(b)** Initial distance $r_0 = 0.8$ pc **(c)** Initial distance $r_0 = 3$ pc

Figure 2: Orbit evolution for different locations

From the plots we can see that as we move away from $M_\bullet$ the behavior of the orbits clearly changes, starting with the amplitude of the movement, since the semi-major axis increases its value. In the case with $r_0 = 0.2pc$ the orbit does not complete a "full precession cycle", while in the other two cases we appreciate even more than one "full precession cycle". Then, we can say that the closer we are from the SMBH, the slower the apsidal precession rate is. We can also observe that the eccentricity of the orbits seems to slightly decrease as we move away from the SMBH.

d) Perturbation theory (again!?). Expand the Hernquist potential to first order in r/s and orbit-average the joint potential

$$\langle\phi\rangle = \langle\phi_\bullet\rangle + \langle\phi_H\rangle \tag{3}$$

using Delaunay variables. Compute the apsidal precession rate of a star $\dot{\omega}$. Discuss your results in light of the experiments in c.

We have

$$\phi_H = -\frac{GM_c}{s+r}$$

And we can rewrite this as:

$$\phi_H = -\frac{GM_c}{s}(1+r/s)^{-1}$$

Expanding this to first order considering $(1+x)^{-1} \approx 1-x$ we get

$$\phi_H = \frac{GM_c(r-s)}{s^2}$$

Then

$$\phi = -\frac{GM_\bullet}{r} + \frac{GM_c r}{s^2} - \frac{GM_c}{s}$$

If we take the orbit-average we have:

$$\begin{aligned}\langle\phi\rangle &= \left\langle -\frac{GM_\bullet}{r}\right\rangle + \left\langle \frac{GM_c r}{s^2}\right\rangle - \left\langle \frac{GM_c}{s}\right\rangle \\ &= -GM_\bullet \left\langle \frac{1}{r}\right\rangle + \frac{GM_c}{s^2}\langle r\rangle - \frac{GM_c}{s}\end{aligned}$$

From the previous unit we know that

$$\left\langle \frac{a}{r}\right\rangle = 1 \quad\Rightarrow\quad \left\langle \frac{1}{r}\right\rangle = \frac{1}{a}$$

and

$$\left\langle \frac{r}{a}\right\rangle = 1 + \frac{e^2}{2} \quad\Rightarrow\quad \langle r\rangle = a + \frac{ae^2}{2}$$

Then we have

$$\langle\phi\rangle = -\frac{GM_\bullet}{a} + \frac{GM_c}{s^2}\left(a + \frac{ae^2}{2}\right) - \frac{GM_c}{s}$$

To write this in terms of the Delaunay variables, we remember that

$$L = \sqrt{GMa} \qquad \mathcal{G} = L\sqrt{1-e^2} \qquad H = \mathcal{G}\cos I$$

In this case we have $M = M_\bullet + M_c$, then we can write:

$$a = \frac{L^2}{G(M_\bullet + M_c)} \qquad e^2 = 1 - \left(\frac{\mathcal{G}}{L}\right)^2$$

Replacing these values in the equation for $\langle \phi \rangle$ we have:

$$
\begin{aligned}
H_{sec} &= -\frac{GM_\bullet}{\frac{L^2}{G(M_\bullet + M_c)}} + \frac{GM_c}{s^2}\left(\frac{L^2}{G(M_\bullet + M_c)} + \frac{\frac{L^2}{G(M_\bullet+M_c)}\left(1-\left(\frac{\mathcal{G}}{L}\right)^2\right)}{2}\right) - \frac{GM_c}{s} \\
&= -\frac{G^2 M_\bullet (M_\bullet + M_c)}{L^2} + \frac{M_c L^2}{s^2(M_\bullet + M_c)} + \frac{M_c L^2}{2s^2(M_\bullet + M_c)} - \frac{M_c L^2}{2s^2(M_\bullet + M_c)}\left(\frac{\mathcal{G}}{L}\right)^2 - \frac{GM_c}{s} \\
&= -\frac{G^2 M_\bullet (M_\bullet + M_c)}{L^2} + \frac{M_c}{2s^2(M_\bullet + M_c)}\left(3L^2 - \mathcal{G}^2\right) - \frac{GM_c}{s}
\end{aligned}
$$

Now we use Hamilton equations to compute $\dot{\omega}$:

$$
\begin{aligned}
\dot{\omega} &= \frac{\partial H_{sec}}{\partial \mathcal{G}} \\
&= -\frac{M_c \mathcal{G}}{s^2(M_\bullet + M_c)}
\end{aligned}
$$

Replacing $\mathcal{G} = \sqrt{G(M_\bullet + M_c)a(1-e^2)}$ we have:

$$
\begin{aligned}
\dot{\omega} &= -\frac{M_c\sqrt{G(M_\bullet + M_c)a(1-e^2)}}{s^2(M_\bullet + M_c)} \\
&= -\frac{M_c}{s^2}\sqrt{\frac{Ga(1-e^2)}{M_\bullet + M_c}}
\end{aligned}
$$

If we consider $M_c = 8M_\bullet$ then

$$
\dot{\omega} = -\frac{8}{3}\frac{\sqrt{GM_\bullet a(1-e^2)}}{s^2}
$$

This expression tell us that further away from the SMBH, the apsidal precession rate is faster, since its value increases due to the increment in a. This result completely agrees with what we concluded from the orbit plots for different locations. The minus sign in front of the expression means that there is a retrograde precession, which we could have inferred from Figure 2a.

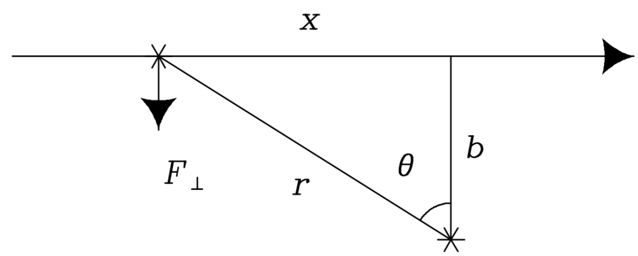


Figure 3: Graphic representation of the problem

Problem 3: Two-body relaxation

a) Follow the steps in section 1.2.1 of BT to compute the two-body relaxation time t_{relax}**.**

We are asked to compute the two-body relaxation time, which refers to the time it takes for a star's orbit to be deflected by another star

In order to do this, we define two bodies: the subject star (the one that gets deflected) and the field star (which causes the deflection). We have to estimate the amount ($\delta\vec{v}$) by which the encounter between the two stars deflects the velocity ($\vec{v}$) of the subject star. We assume that the field star is stationary during the encounter and that

$$\frac{|\delta\vec{v}|}{v} \ll 1$$

We also take $\delta v \perp v$, since the accelerations parallel to v average to zero. We also assume that the subject star passes the field star on a straight-line trajectory and we estimate the magnitude of the velocity change ($\delta v = |\delta\vec{v}|$) by integrating the perpendicular force $F_\perp$ along all the trajectory of the subject star. We have that:

$$F_\perp = \frac{Gm^2}{r} cos\theta = \frac{Gm^2}{b^2 + x^2} cos\theta$$

But using Fig.3 geometry, we can write

$$cos\theta = \frac{b}{r} = \frac{b}{\sqrt{b^2 + x^2}}$$

Then

$$F_\perp = \frac{Gm^2 b}{(b^2 + x^2)^{3/2}}$$

By Newton's second law we have

$$\vec{F} = m\dot{\vec{v}}$$

So

$$\delta v = \frac{1}{m} \int_{-\infty}^{\infty} F_\perp dt$$

Since the subject star moves along the x axis, $x = vt$, and we write

$$F_{\perp} = \frac{Gm^2}{b^2}\left[1 + \left(\frac{vt}{b}\right)^2\right]^{-3/2}$$

Replacing this in the expression for δv we get

$$\delta v = \frac{Gm}{b^2}\int_{-\infty}^{\infty} \frac{dt}{\left[1+\left(\frac{vt}{b}\right)^2\right]^{3/2}}$$

Changing variables to $s = vt/b$ so that $ds = vdt/b$ we have

$$\begin{aligned}
\delta v &= \frac{Gm}{b^2}\int_{-\infty}^{\infty} \frac{b\ ds}{v\left[1+s^2\right]^{3/2}} \\
&= \frac{Gm}{bv}\int_{-\infty}^{\infty} \frac{ds}{\left[1+s^2\right]^{3/2}} \\
&= \frac{2Gm}{bv}\int_{0}^{\infty} \frac{ds}{\left[1+s^2\right]^{3/2}} \\
&= \frac{2Gm}{bv}\left.\frac{s}{\sqrt{s^2+1}}\right|_0^{\infty} \to 1 \\
&= \frac{2Gm}{bv}
\end{aligned}$$

We must notice that when $\delta v \simeq v$, then our expression does not work out, since our assumption of a straight-line trajectory breaks down. This occurs if $b \lesssim b_{90} \equiv 2Gm/v^2$.

Considering impact parameters within the range $[b, b+db]$, the number of encounters that the subject star suffers while it crosses a galaxy of radius R with N stars ($\Sigma = \frac{N}{\pi R^2}$) is given by

$$\delta n = \Sigma\ dA = \frac{N}{\pi R^2}2\pi b db = \frac{2N}{R^2}bdb$$

Each encounter changes the velocity of the subject star by an amount $\delta\vec{v}$, but these perturbations are randomly oriented so at the end the average perturbation would be zero. Nevertheless, the mean-square change in velocity is not zero:

$$\Sigma\delta v^2 \simeq \delta v^2 \delta n = \left(\frac{2Gm}{bv}\right)^2 \frac{2N}{R^2}bdb$$

Integrating this expression through all the range of b values we have:

$$\Delta v^2 = \int_{-\infty}^{\infty} \Sigma\delta v^2 = \int_{-\infty}^{\infty}\left(\frac{2Gm}{bv}\right)^2 \frac{2N}{R^2}bdb = 8N\left(\frac{Gm}{Rv}\right)^2\int_{-\infty}^{\infty}\frac{db}{b}$$

Since if $b < b_{90}$ our assumption of a straight-line trajectory breaks down, we set $b_{min} = b_{90}$. Moreover, our assumption of a homogeneous distribution field stars breaks down for impact parameters of order R, so we set $b_{max} = R$. Then

$$\Delta v^2 = 8N\left(\frac{Gm}{Rv}\right)^2\int_{b_{min}}^{b_{max}}\frac{db}{b} = 8N\left(\frac{Gm}{Rv}\right)^2 ln(b)\Big|_{b_{min}}^{b_{max}} = 8N\left(\frac{Gm}{Rv}\right)^2 ln\left(\frac{b_{max}}{b_{min}}\right)$$

We define:

$$\Lambda = \frac{b_{max}}{b_{min}} = \frac{R}{b_{90}} = \frac{Rv^2}{2Gm}$$

Then

$$\Delta v^2 = 8N \left(\frac{Gm}{Rv}\right)^2 ln\Lambda$$

We know that the typical speed of a star is equivalent to the speed of a particle in a circular orbit at the edge of the galaxy:

$$v^2 \approx \frac{GNm}{R}$$

So

$$R \approx \frac{GNm}{v^2}$$

And then we have

$$\Delta v^2 = 8N \left(\frac{v}{N}\right)^2 ln\Lambda$$

Which can be written as

$$\frac{\Delta v^2}{v^2} \approx \frac{8ln\Lambda}{N}$$

The subject star's orbit would be deflected by an amount Δv^2 after each crossing of the galaxy. The number of encounters required to $\Delta v^2 \approx v$ is given by

$$n_{relax} \simeq \frac{N}{8ln\Lambda}$$

We have to notice that

$$N \approx \frac{Rv^2}{Gm} \qquad \& \qquad \Lambda = \frac{Rv^2}{2Gm}$$

So given the orders of magnitude we are working with, we can say $N \approx \Lambda$ and then

$$n_{relax} \simeq \frac{N}{8lnN}$$

Now, we define t_{cross} as the time needed for a typical star to cross the galaxy once ($t_{cross} = R/v$) so that the relaxation time is given by:

$$t_{relax} = n_{relax} t_{cross} \simeq \frac{0.1N}{lnN} t_{cross}$$

b) Provide with an order-of-magnitude estimate of t_{relax} in the Solar neighborhood, globular clusters, and the galactic center ($r \sim 0.3$ pc). You can look up the typical numbers, but just be explicit on what you assume.

- **Globular Clusters**

According to BT 1.1.4, Globular Clusters contain $N \sim 10^5$ stars and as can be extracted from Fig. 4, $t_{cross} = 1Myr$. Then replacing these values in our final expression for t_{relax} we have:

$$t_{relax}^{GC} \approx \frac{0.1 \cdot 10^5}{ln(10^5)} \cdot 10^5 yr \approx 9 \cdot 10^7 \ yr$$

- **Solar Neighborhood (Galaxy)**

For the solar neighborhood we consider $R \sim 10kpc$ and the data from Fig. 5. We assume the RMS velocity of nearby stars to be $\approx 50km/s$. So that

$$t_{cross} \approx \frac{10kpc}{50km \ s^{-1}} \approx 2 \cdot 10^8 \ yr$$

If we assume that $1M_\odot$ is the typical mass of the stars in our solar neighborhood and consider the number of stars in our galaxy $N \sim 10^{11}$ then

$$t_{relax}^{Solar} \approx \frac{0.1N}{lnN} t_{cross} \approx 8 \cdot 10^7 \ Gyr$$

- **Galactic Center**

For the Galactic center we consider $R \approx 0.3pc$ and $v \approx 100km/s$[1]. Then

$$t_{cross} \approx \frac{0.3pc}{100km \ s^{-1}} \approx 3 \cdot 10^3 \ yr$$

Again, if we assume that $1M_\odot$ is the typical mass of the stars we have

$$N \approx \frac{Rv^2}{Gm} \approx 7 \cdot 10^5$$

$$t_{relax}^{GalacticCenter} = n_{relax} t_{cross} \simeq \frac{0.1N}{lnN} t_{cross} \approx 1.5 \cdot 10^7 \ yr$$

[1] See circular speed curve of the galaxy on BT Fig. 2.20

Table 1.3 Parameters of globular and open clusters

	globular	open
central density ρ_0	$1 \times 10^4\, \mathcal{M}_\odot\, \mathrm{pc}^{-3}$	$10\, \mathcal{M}_\odot\, \mathrm{pc}^{-3}$
core radius r_c	1 pc	1 pc
half-mass radius r_h	3 pc	2 pc
tidal radius r_t	35 pc	10 pc
central velocity dispersion σ_0	$6\, \mathrm{km\, s^{-1}}$	$0.3\, \mathrm{km\, s^{-1}}$
crossing time r_h/σ_0 (line-of-sight)	0.5 Myr	7 Myr
mass-to-light ratio Υ_R	$2\Upsilon_\odot$	$1\Upsilon_\odot$
mass M	$2 \times 10^5\, \mathcal{M}_\odot$	$300\, \mathcal{M}_\odot$
lifetime	10 Gyr	300 Myr
number in the Galaxy	150	10^5

NOTES: Values for globular clusters are medians from the compilation of Harris (1996). Values for open clusters are from Figure 8.5, Piskunov et al. (2007), and other sources.

Figure 4: Parameters of globular and open clusters from BT

Problem 4: Orbits in the Solar Neighborhood

We shall assume a simple model for the stellar disk in our galaxy following Miyamoto & Nagai:

$$\phi_{\mathrm{MN}}(R, z) = -\frac{GM}{\sqrt{R^2 + (a + \sqrt{z^2 + b^2})^2}} \tag{4}$$

with $M = 6.8 \times 10^{10} M_\odot$, $a = 3$ kpc and $b = 280$ pc[2].

We place the guiding center at $R_g = 8$ kpc and study the motion of stars, including our Sun, in its vicinity using the epicyclic approximation.

a) Solve for the motion of the guiding center and compute $\Omega(R_g)$, L_z and the circular velocity v_{circ} in km s^{-1}.

We know that

$$\Omega^2(R) = \frac{1}{R}\left(\frac{\partial \phi}{\partial R}\right)_{(R,0)}$$

Given our potential we have

$$\begin{aligned}
\left(\frac{\partial \phi}{\partial R}\right) &= \frac{\partial}{\partial R}\left(-\frac{GM}{\sqrt{R^2 + (a + \sqrt{z^2 + b^2})^2}}\right) \\
&= \frac{GM}{2\left(R^2 + (a + \sqrt{z^2 + b^2})^2\right)^{3/2}} 2R \\
&= \frac{GMR}{\left(R^2 + (a + \sqrt{z^2 + b^2})^2\right)^{3/2}}
\end{aligned}$$

[2] gp.MiyamotoNagaiPotential(m=6.8E10*u.Msun, a=3*u.kpc, b=280*u.pc,units=galactic)

Table 1.2 Properties of the Galaxy

Global properties:	
disk scale length $R_{\rm d}$	$(2.5 \pm 0.5)\,\mathrm{kpc}$
disk luminosity	$(2.5 \pm 1) \times 10^{10}\,L_\odot$
bulge luminosity	$(5 \pm 2) \times 10^{9}\,L_\odot$
total luminosity	$(3.0 \pm 1) \times 10^{10}\,L_\odot$
disk mass	$(4.5 \pm 0.5) \times 10^{10}\,\mathcal{M}_\odot$
bulge mass	$(4.5 \pm 1.5) \times 10^{9}\,\mathcal{M}_\odot$
dark halo mass	$(2\,^{+3}_{-1.8}) \times 10^{12}\,\mathcal{M}_\odot$
dark halo half-mass radius	$(100\,^{+100}_{-80})\,\mathrm{kpc}$
disk mass-to-light ratio Υ_R	$(1.8 \pm 0.7)\Upsilon_\odot$
total mass-to-light ratio Υ_R	$(70\,^{+100}_{-63})\Upsilon_\odot$
black-hole mass	$(3.9 \pm 0.3) \times 10^{6}\,\mathcal{M}_\odot$
Hubble type	Sbc
Solar neighborhood properties:	
solar radius R_0	$(8.0 \pm 0.5)\,\mathrm{kpc}$
circular speed v_0	$(220 \pm 20)\,\mathrm{km\,s^{-1}}$
angular speed: from v_0/R_0	$(27.5 \pm 3)\,\mathrm{km\,s^{-1}\,kpc^{-1}}$
from Sgr A*	$(29.5 \pm 0.2)\,\mathrm{km\,s^{-1}\,kpc^{-1}}$
disk density ρ_0	$(0.09 \pm 0.01)\,\mathcal{M}_\odot\,\mathrm{pc^{-3}}$
disk surface density Σ_0	$(49 \pm 6)\,\mathcal{M}_\odot\,\mathrm{pc^{-2}}$
disk thickness Σ_0/ρ_0	$500\,\mathrm{pc}$
scale height $z_{\rm d}$ (old stars)	$300\,\mathrm{pc}$
rotation period $2\pi/\Omega_0$	$(220 \pm 30)\,\mathrm{Myr}$
vertical frequency $\nu_0 = \sqrt{4\pi G\rho_0}$	$(2.3 \pm 0.1) \times 10^{-15}\,\mathrm{Hz}$
	$= (70 \pm 4)\,\mathrm{km\,s^{-1}\,kpc^{-1}}$
vertical period $2\pi/\nu_0$	$87\,\mathrm{Myr}$
Oort's A constant	$(14.8 \pm 0.8)\,\mathrm{km\,s^{-1}\,kpc^{-1}}$
Oort's B constant	$-(12.4 \pm 0.6)\,\mathrm{km\,s^{-1}\,kpc^{-1}}$
epicycle frequency $\kappa_0 = \sqrt{-4B(A-B)}$	$(37 \pm 3)\,\mathrm{km\,s^{-1}\,kpc^{-1}}$
radial dispersion of old stars	$(38 \pm 2)\,\mathrm{km\,s^{-1}}$
vertical dispersion of old stars	$(19 \pm 2)\,\mathrm{km\,s^{-1}}$
RMS velocity of old stars	$(50 \pm 3)\,\mathrm{km\,s^{-1}}$
escape speed $v_{\rm e}(R_0)$	$(550 \pm 50)\,\mathrm{km\,s^{-1}}$

NOTES: See §2.7 and §§10.1 and 10.3 of BM for more detail. Luminosities are in the R band at $\lambda = 660\,\mathrm{nm}$. The halo mass and half-mass radius are taken from Wilkinson & Evans (1999). The angular speed of the central black hole (Sgr A*) relative to an extragalactic frame is from Reid & Brunthaler (2004). The density in the midplane of the disk, ρ_0, and the surface density Σ_0 are taken from Table 1.1. The scale height $z_{\rm d}$ is defined by equation (1.10). The RMS velocity of old stars is the square root of the sum of the squared dispersions along the three principal axes of the velocity-dispersion tensor (BM Table 10.2). Escape speed is from Smith et al. (2007).

Figure 5: Parameters for the solar neighborhood from BT

Replacing with $R = R_g$ and $z = 0$ we get

$$\Omega^2(R_g) = \frac{1}{R_g} \frac{GMR_g}{\left(R_g^2 + (a+b)^2\right)^{3/2}}$$

and then

$$\boxed{\Omega(R_g) = \sqrt{\frac{GM}{\left(R_g^2 + (a+b)^2\right)^{3/2}}}}$$

On the other hand we know

$$L_z^2 = R^3 \left(\frac{\partial \phi}{\partial R}\right)_{(R,0)} = R^4\ \Omega(R)^2$$

So replacing with $R = R_g$ we have

$$\boxed{L_z = \sqrt{\frac{GMR_g^4}{\left(R_g^2 + (a+b)^2\right)^{3/2}}}}$$

Finally,

$$v_c = \Omega(R)R$$

so that

$$\boxed{v_c = R_g \sqrt{\frac{GM}{\left(R_g^2 + (a+b)^2\right)^{3/2}}}}$$

Using the values given above ($R_g = 8$ kpc, $M = 6.8 \cdot 10^{10} M_\odot$, $a = 3$ kpc and $b = 280$ pc) we have:

$$\boxed{\begin{aligned} \Omega(R_g) &= 6.89 \cdot 10^{-16}\ s^{-1} \\ L_z &= 4.2 \cdot 10^{19}\ km^2\ s^{-1} = 1.39\ pc^2\ yr^{-1} \\ v_c &= 170.17\ km\ s^{-1} \end{aligned}}$$

b) Integrate the motion of the guiding center using Gala for one azimuthal period $T_\psi = 2\pi/\Omega(R_g)$. Check that the motion is indeed circular.

In order to integrate the motion of the guiding center using Gala, we need to define initial conditions. We know that the distance is $R = R_g$, so we can set the initial position as $(R_g, 0, 0)$. At this position in the orbit we know that the initial velocity must be $(0, v_c, 0)$. Then, we integrate for $T_\psi = 2\pi/\Omega(R_g) = 288.83$ Myr and we obtain the orbit showed in Figure 6, which clearly indicates that the motion is indeed circular.

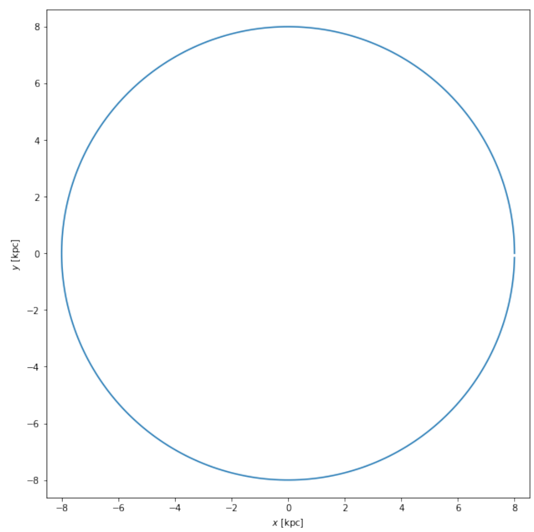


Figure 6: Integrated orbit for one azimuthal period considering $R = R_g$

c) Find the epicyclic frequency κ and vertical frequency ν at R_g.
We know that according to Eq. (3.79a) of BT

$$\kappa^2(R_g) = \left(\frac{\partial^2 \phi}{\partial R^2}\right)_{(R_g,0)} + \frac{3}{R_g}\left(\frac{\partial \phi}{\partial R}\right)_{(R_g,0)}$$

and Eq. (3.79b)

$$\nu^2(R_g) = \left(\frac{\partial^2 \phi}{\partial z^2}\right)_{(R_g,0)}$$

then we have

$$\begin{aligned}
\kappa^2(R_g) &= \frac{\partial}{\partial R}\left(\frac{GMR}{(R^2+(a+b)^2)^{3/2}}\right) + \frac{3GM}{(R^2+(a+b)^2)^{3/2}}\bigg|_{R=R_g} \\
&= \frac{GM}{(R^2+(a+b)^2)^{3/2}} - \frac{3GMR^2}{(R^2+(a+b)^2)^{5/2}} + \frac{3GM}{(R^2+(a+b)^2)^{3/2}}\bigg|_{R=R_g} \\
&= \frac{GM(4(a+b)^2+R_g^2)}{(R_g^2+(a+b)^2)^{5/2}}
\end{aligned}$$

and

$$\begin{aligned}
\nu^2(R_g) &= \frac{\partial}{\partial z}\ \frac{GMz\ (a+\sqrt{z^2+b^2})}{(R_g^2+(a+\sqrt{z^2+b^2})^2)^{3/2}\ \sqrt{(z^2+b^2)}}\bigg|_{z=0} \\
&= \frac{GM(a+b)}{b\ (R_g^2+(a+b)^2)^{3/2}}
\end{aligned}$$

Evaluating both expressions we obtain:

$$\boxed{\begin{aligned}\kappa(R_g) &= 8.248\cdot 10^{-16}\ s^{-1}\\ \nu(R_g) &= 2.359\cdot 10^{-15}\ s^{-1}\end{aligned}}$$

d) We shall compare the results from the epicyclic approximation with direct integrations. To do this integrate the orbits with Gala setting $R = R_g$ and $z = 100$ pc, and velocities $v_x = 5$ km s^{-1}, $v_y = v_{\rm circ}$, and $v_z = 0$. Use cylindrical coordinates to plot the motion[3].

Compare with the epicyclic motion $x \equiv R - R_g = x_0 \sin(\kappa t)$ and $z = z_0 \cos(\nu t)$ with $z_0 = 100$ pc. Set x_0 to match the integration results.

Integrating the orbits as defined above, we get the blue curve on Fig. 7. If we compare this result with the epicyclic motion considering $x_0 = 0.2\ kpc$ so that integration results match with the epicyclic approximation, we obtain the orange curve.

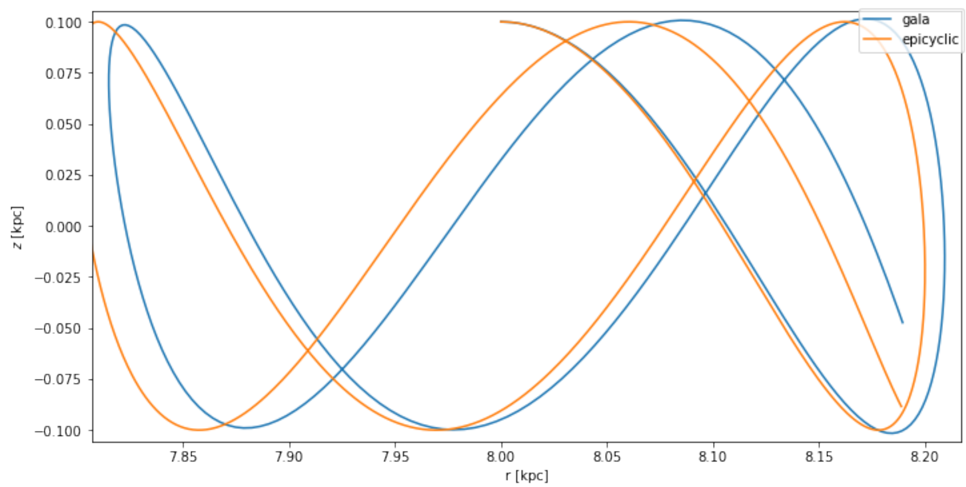


Figure 7: Comparison between epicyclic approximation and direct integrations of the orbit considering $v_x = 5kms^{-1}$

We can notice that both curves differs from each other mainly on the radial axis, since the epicyclic motion yields a broader range of r values in relation to direct integration's orbit, which is also a little asymmetric. Nevertheless, we can state that the epicyclic approximation works pretty fine in this case.

e) Repeat the integration in (d) but setting $v_x = 200$ km s^{-1}. Would the approximation work here? Comment.

If we change the initial velocity on the x axis, then the relation between direct integrations and the epicyclic approximation changes drastically as we can see on Fig. 8

[3]fig = orbit.represent_as('cylindrical').plot(['rho', 'z'])

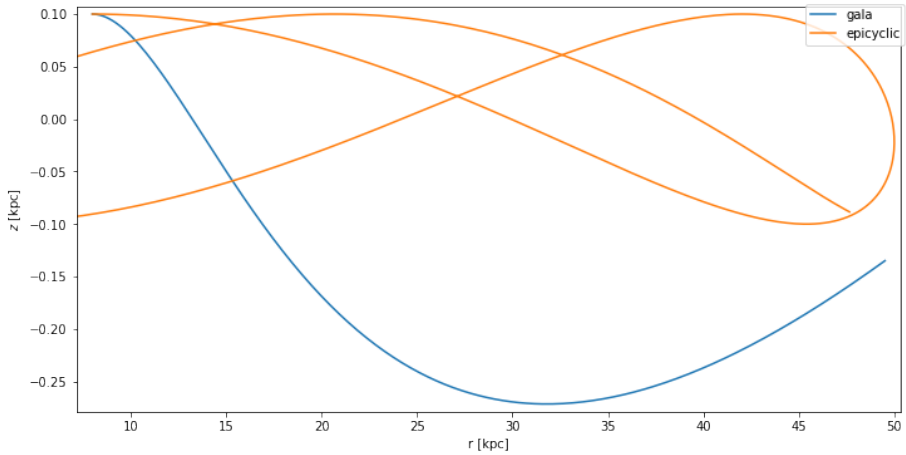


Figure 8: Comparison between epicyclic approximation and direct integrations of the orbit considering $v_x = 200kms^{-1}$

We can attribute this difference between both resulting orbits to the initial velocity that we considered, since it must be higher than the escape velocity for this particular potential. This means that the star is no longer bounded to the galaxy, which explains the shape of the orbit obtained with Gala. In fact, if we compute the escape velocity we have:

$$v_e = \sqrt{2|\phi|} = \sqrt{2\left|-\frac{GM}{\sqrt{R^2 + (a + \sqrt{z^2 + b^2})^2}}\right|} \approx 260\ km\ s^{-1}$$

And the initial velocity we considered was

$$v_0 = \sqrt{v_x^2 + v_y^2} \approx 263\ km\ s^{-1}$$

which is slightly higher than the escape velocity.

Problem 5: razor-thin disk

Compute the potential Φ of a razor-thin disk with surface density Σ at $z = 0$.

In order to compute the potential of a razor-thin disk with surface density Σ at $z = 0$, we can use Gauss's theorem

$$4\pi G \int_V d^3\mathbf{x}\rho = \int_S d^2S \cdot \nabla\phi$$

Where we can consider the density as

$$\rho(R, z) = \Sigma(R)\zeta(z)$$

To evaluate the integrals, we need to picture a Gaussian surface. For example, we can take a cylinder of height h, radius R and basal area A. If we consider a razor-thin disk, making $h \to 0$, then we have

$$\rho(R, 0) = \Sigma(R)$$

and then

$$4\pi G \int d^3\mathbf{x}\rho = 4\pi G \int \Sigma(R) R dR d\phi$$

Since we know that the surface density is constant, the left side of our expression becomes

$$4\pi G \int d^3\mathbf{x}\rho = 4\pi G\Sigma A$$

On the other hand, for the integral of the right side of our expression we have to integrate over the surface of our cylinder, which, when taking $h \to 0$, translates into:

$$\int d^2S \cdot \nabla\phi = A\nabla\phi\Big|_{z=0^+} - A\nabla\phi\Big|_{z=0^-}$$

Our potential is chosen such that it is axisymmetric and does not depend on z, so we can say that the contribution to the surface integrand at constant R cancel each other (or that $\frac{\partial\phi}{\partial R} \to 0$ when evaluating over z). Therefore, we have

$$\int d^2S \cdot \nabla\phi = A\frac{\partial\phi}{\partial z}\Big|_{z=0^+} - A\frac{\partial\phi}{\partial z}\Big|_{z=0^-} = 2A\left|\frac{\partial\phi}{\partial z}\right|_{z=0}$$

And then equating both sides we find

$$\begin{aligned} 2A\left|\frac{\partial\phi}{\partial z}\right|_{z=0} &= 4\pi G\Sigma A \\ \left|\frac{\partial\phi}{\partial z}\right|_{z=0} &= 2\pi G\Sigma \end{aligned}$$

Integrating over z we obtain

$$\boxed{\begin{aligned} |\phi| &= 2\pi G\Sigma z + C \\ \phi &= 2\pi G\Sigma|z| + C \end{aligned}}$$

Problem 6: radial vs azimuthal period

Using the epicycle approximation, prove that the azimuthal angles $\Delta\psi$ between successive pericenters lies in the range $\pi \leq \psi \leq 2\pi$ in the gravitational field arising from any spherical mass distribution in which the density decreases outwards.

We know that the azimuthal period can be expressed as

$$T_\psi = \frac{2\pi}{|\Delta\psi|} T_r$$

And that using the epicycle approximation we found

$$T_r = \frac{2\pi}{\kappa} \quad ; \quad T_\psi = \frac{2\pi}{\Omega}$$

Then, we can express the azimuthal angles $\Delta\phi$ as

$$|\Delta\psi| = 2\pi\frac{\Omega}{\kappa}$$

where

$$\Omega^2(R) = \frac{1}{R}\left(\frac{\partial\phi}{dR}\right)_{(R,0)}$$

$$\kappa^2(R) = R\frac{d\Omega^2}{dR} + 4\Omega^2$$

If the density decreases outwards then

$$\rho(R) \to 0 \quad if \quad R \to \infty$$

A density distribution function that satisfy this is

$$\rho(R) = \rho_0 R^{-\alpha}$$

With $\alpha \in [0, 3]$, since otherwise the mass enclosed within a radius r diverges at large r. The limit case $\alpha \to 3$ corresponds to a Keplerian Potential.

We can obtain the corresponding potential using Poisson's equation, considering a spherically symmetric potential ($\phi = \phi(R)$)

$$\begin{aligned}
\nabla^2\phi &= 4\pi G\rho \\
\frac{\partial^2\phi}{\partial R^2} &= 4\pi G\rho_0 R^{-\alpha} \\
\phi &= 4\pi G\rho_0 \frac{R^{-\alpha+2}}{(-\alpha+2)(-\alpha+1)}
\end{aligned}$$

Then

$$\phi \propto R^{2-\alpha}$$

so that

$$\begin{aligned}
\Omega^2 &\propto (2-\alpha)R^{-\alpha} \\
\kappa^2 &\propto -\alpha(2-\alpha)R^{-\alpha} + 4(2-\alpha)R^{-\alpha} = (2-\alpha)(4-\alpha)R^{-\alpha}
\end{aligned}$$

and

$$\frac{\Omega}{\kappa} = \sqrt{\frac{1}{4-\alpha}}$$

Now, we must constrain our potential. We know that far away from the center of the mass distribution, the potential could be approximated by a Keplerian potential, such that $\alpha = 3$ and then $\frac{\Omega}{\kappa} = 1$. On the other hand, we know that near the center the circular speed should increase approximately linearly with radius ($v_c = R\,\Omega(R)$), which means that the circular frequency must remain almost constant. In our case, this implies that $\alpha \to 0$ and then $\frac{\Omega}{\kappa} \to \frac{1}{2}$. Considering these boundary conditions, we have

$$\boxed{\pi \leq |\Delta\psi| \leq 2\pi}$$

Problem 7: epicyclic frequencies

Compute the ratio κ/Ω for:

From BT 3.2.3 we know

$$\Omega^2(R) = \frac{1}{R}\left(\frac{\partial\phi}{dR}\right)_{(R,0)}$$

$$\kappa^2(R) = R\frac{d\Omega^2}{dR} + 4\Omega^2$$

Then:

- **the Keplerian potential $\phi \propto R^{-1}$** In this case we have

$$\phi = cR^{-1}$$

Then

$$\Omega^2(R) = -cR^{-3}$$

And

$$\kappa^2(R) = 3cR^{-3} - 4cR^{-3} = -cR^{-3}$$

So

$$\boxed{\frac{\kappa}{\Omega} = 1}$$

- **the spherical harmonic $\phi \propto R^2$** Here

$$\Omega^2(R) = 2c$$

And

$$\kappa^2(R) = 8c$$

So

$$\boxed{\frac{\kappa}{\Omega} = 2}$$

- **a flat rotation curve $\phi \propto \ln R$**

Finally for a flat rotation curve we have

$$\Omega^2(R) = cR^{-2}$$

And

$$\kappa^2(R) = -2cR^{-2} + 4cR{-}2 = 2cR^{-2}$$

So

$$\boxed{\frac{\kappa}{\Omega} = \sqrt{2} \approx 1.41}$$

As seen in class, the ration κ/Ω for our Solar neighborhood is

$$\frac{\kappa_0}{\Omega_0} = 1.35 \pm 0.05$$

So the **flat rotation curve potential** is the one that fits best the motion in our Solar neighborhood.

Problem 8: Eddington's Formula & Jeans theorem

We shall derive the distribution function for the Plummer potential step by step and compute the dispersion velocity.

a) First, compute the density $\nu = \rho/M$ as a function of $\Psi = -\phi$ and calculate

$$\frac{\mathbf{d^2}\nu}{\mathbf{d\Psi^2}} \tag{5}$$

We know that for the Plummer model we have

$$\phi = -\frac{GM}{\sqrt{r^2 + b^2}} \tag{6}$$

and

$$\rho(r) = \frac{3M}{4\pi b^3}\left(1 + \frac{r^2}{b^2}\right)^{-5/2} \tag{7}$$

Then in this case

$$\Psi = \frac{GM}{\sqrt{r^2 + b^2}} \tag{8}$$

We need to express $\rho(r)$ as a function of Ψ. From Eq. 8 we know

$$r^2 = \left(\frac{GM}{\Psi}\right)^2 - b^2$$

Replacing this into Eq. 7 we have:

$$\begin{aligned}
\rho(\Psi) &= \frac{3M}{4\pi b^3}\left(1 + \frac{\left(\frac{GM}{\Psi}\right)^2 - b^2}{b^2}\right)^{-5/2} \\
&= \frac{3M}{4\pi b^3}\left(\frac{\cancel{b^2} + \left(\frac{GM}{\Psi}\right)^2 \cancel{- b^2}}{b^2}\right)^{-5/2} \\
&= \frac{3M}{4\pi b^3}\left(\left(\frac{GM}{\Psi b}\right)^2\right)^{-5/2} \\
&= \frac{3M}{4\pi b^3}\left(\frac{\Psi b}{GM}\right)^5
\end{aligned}$$

Then

$$\boxed{\nu(\Psi) = \frac{\rho}{M} = \frac{3}{4\pi b^3}\left(\frac{\Psi b}{GM}\right)^5}$$

And we have

$$\begin{aligned}
\frac{d^2\nu}{d\Psi^2} &= \frac{d}{d\Psi}\left(\frac{d}{d\Psi}\frac{3}{4\pi b^3}\left(\frac{\Psi b}{GM}\right)^5\right) \\
&= \frac{d}{d\Psi}\left(\frac{15\Psi^4}{4\pi b^3}\left(\frac{b}{GM}\right)^5\right) \\
&= \frac{15\Psi^3}{\pi b^3}\left(\frac{b}{GM}\right)^5
\end{aligned}$$

So

$$\boxed{\frac{d^2\nu}{d\Psi^2} = \frac{15\Psi^3 b^2}{\pi(GM)^5}}$$

b) Use Equation 4.46b in BT to compute the distribution function $f_P(\mathcal{E})$ where $\mathcal{E} = \Psi - v^2/2$. Note that the result could be generalized for polytropic models (see Eq. 4.83 of BT)

The equation 4.46b in BT is:

$$f_P(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\left[\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}}\frac{d^2\nu}{d\Psi^2} + \frac{1}{\sqrt{\mathcal{E}}}\left(\frac{d\nu}{d\Psi}\right)_{(\Psi=0)}\right]$$

Given that

$$\left(\frac{d\nu}{d\Psi}\right)_{(\Psi=0)} = \frac{15\Psi^4}{4\pi b^3}\left(\frac{b}{GM}\right)^5\Bigg|_{\Psi=0} = 0$$

We have

$$f_P(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}}\frac{d^2\nu}{d\Psi^2}$$

Replacing $\frac{d^2\nu}{d\Psi^2}$ we get

$$f_P(\mathcal{E}) = \frac{15b^2}{\sqrt{8}\pi^3(GM)^5}\int_0^{\mathcal{E}} \frac{\Psi^3 d\Psi}{\sqrt{\mathcal{E}-\Psi}}$$

Solving the integral with Wolfram

$$\boxed{f_P(\mathcal{E}) = \frac{15b^2}{\sqrt{8}\pi^3(GM)^5}\frac{32\mathcal{E}^{7/2}}{35} = \frac{24\sqrt{2}}{7\pi^3}\frac{b^2}{(GM)^5}\mathcal{E}^{7/2}}$$

We can notice that this result could be written as $f_P(\mathcal{E}) = F\mathcal{E}^{7/2}$. Following Eq. 4.83 of BT and noting that the density previously calculated rises as $\nu \propto \Psi^5$, then we can write $f_P(\mathcal{E}) = F\mathcal{E}^{5-3/2}$. We can also note that if $\Psi < 0$, then $\nu < 0$, which has no sense in our problem. Also if $\Psi < 0$, $\mathcal{E} < 0$, then we can generalize our result for polytropic models as

$$f_P(\mathcal{E}) = \begin{cases} F\mathcal{E}^{n-3/2} & (\mathcal{E} > 0) \\ 0 & (\mathcal{E} \leq 0) \end{cases}$$

c) Calculate the velocity dispersion σ:

$$\sigma^2 = \frac{1}{\nu(r)}\int_0^\infty 4\pi v^4 dv f(r, v).$$

Use the Jeans theorem to independently compute σ^2. Compare your results.

We have

$$\nu(\Psi) = \frac{\rho}{M} = \frac{3}{4\pi b^3}\left(\frac{\Psi b}{GM}\right)^5$$

And

$$f(\mathcal{E}) = \frac{24\sqrt{2}}{7\pi^3}\frac{b^2}{(GM)^5}\mathcal{E}^{7/2}$$

Knowing that

$$\mathcal{E} = \Psi - \frac{v^2}{2}$$

We can rewrite

$$f(\Psi) = \frac{24\sqrt{2}}{7\pi^3}\frac{b^2}{(GM)^5}\left(\Psi - \frac{v^2}{2}\right)^{7/2}$$

So that

$$\begin{aligned}
\sigma^2 &= \frac{4\pi b^3}{3}\left(\frac{GM}{\Psi b}\right)^5 \int_0^{\sqrt{2\Psi}} 4\pi v^4 \frac{24\sqrt{2}}{7\pi^3}\frac{b^2}{(GM)^5}\left(\Psi - \frac{v^2}{2}\right)^{7/2} dv \\
&= \frac{128\sqrt{2}}{7\pi\Psi^5}\int_0^{\sqrt{2\Psi}} v^4 \left(\Psi - \frac{v^2}{2}\right)^{7/2} dv \\
&= \frac{128\sqrt{2}}{7\pi\Psi^5}\frac{7\pi\Psi^6}{256\sqrt{2}} \\
&= \frac{GM}{2\sqrt{r^2+b^2}}
\end{aligned}$$

If $\sigma_r^2 = \sigma_\theta^2 = \sigma_\phi^2$, then $\sigma^2 = 3\sigma_r^2$ and we have

$$\sigma_r^2 = \frac{GM}{6\sqrt{r^2+b^2}}$$

While with Jeans theorem the procedure is much simpler. From Eq. 4.216 of BT we have

$$\sigma_r^2 = \frac{1}{r^{2\beta}\nu(r)}\int_r^\infty dr' r'^{2\beta}\nu(r')\frac{d\Phi}{dr'}$$

And given our potential we know

$$\frac{d\Phi}{dr} = \frac{GMr}{(r^2+b^2)^{3/2}}$$

We need to express $\nu(\Psi)$ as $\nu(r)$, so:

$$\nu(r) = \frac{3}{4\pi b^3}\left(\frac{b}{\sqrt{r^2+b^2}}\right)^5$$

And then considering $\beta = 0$:

$$\begin{aligned}
\sigma_r^2 &= \frac{1}{\nu(r)}\int_r^\infty dr'\nu(r')\frac{d\Phi}{dr'} \\
&= \frac{4\pi b^3}{3}\left(\frac{\sqrt{r^2+b^2}}{b}\right)^5 \int_r^\infty dr' \frac{3}{4\pi b^3}\left(\frac{b}{\sqrt{r'^2+b^2}}\right)^5 \frac{GMr'}{(r'^2+b^2)^{3/2}} \\
&= GM\left(\sqrt{r^2+b^2}\right)^5 \int_r^\infty \frac{r'}{(r'^2+b^2)^4} dr' \\
&= GM\left(\sqrt{r^2+b^2}\right)^5 \frac{1}{6(r^2+b^2)^3} \\
&= \frac{GM}{6\sqrt{r^2+b^2}}
\end{aligned}$$

Which is the same result we obtained above.

d) Let us assume that there is a intermediate-mass black hole of mass M_{IMBH} in the center of the globular cluster (i.e., Plummer potential). Compute the resulting dispersion velocity using the Jeans theorem[4].

Assuming that there is a black hole in the center of the globular cluster, we can compute the resulting dispersion velocity as:

$$\sigma_r^2 = \frac{1}{\nu(r)} \int_r^\infty \frac{GM_{enc}(r')\nu(r')}{r'^2} dr'$$

Given that for Plummer potential

$$M_{enc}(r) = \frac{Mr^3}{(r^2+b^2)^{3/2}}$$

If we include the mass of the black hole we have[5]

$$\begin{aligned}
\sigma_r^2 &= \frac{G}{\nu(r)} \int_r^\infty \frac{\left(\frac{Mr'^3}{(r'^2+b^2)^{3/2}} + M_{IMBH}\right)\nu(r')}{r'^2} dr' \\
&= \frac{4\pi G b^3}{3}\left(\frac{\sqrt{r'^2+b^2}}{b}\right)^5 \left(\int_r^\infty \frac{Mr^3}{(r'^2+b^2)^{3/2}} \frac{3}{4\pi b^3}\left(\frac{b}{\sqrt{r^2+b^2}}\right)^5 \frac{dr'}{r'^2} + \int_r^\infty M_{IMBH}\frac{3}{4\pi b^3}\left(\frac{b}{\sqrt{r'^2+b^2}}\right)^5 \frac{dr'}{r'^2}\right) \\
&= G\left(\sqrt{r^2+b^2}\right)^5 \left(M\int_r^\infty \frac{r'}{(r'^2+b^2)^4} dr' + M_{IMBH}\int_r^\infty \left(\frac{1}{\sqrt{r'^2+b^2}}\right)^5 \frac{dr'}{r'^2}\right) \\
&= \frac{GM}{6\sqrt{r^2+b^2}} + GM_{IMBH}\left(\frac{3b^4 + 4b^2 r(3r - 2\sqrt{b^2+r^2}) + 8r^3(r-\sqrt{b^2+r^2})}{3b^6 r(b^2+r^2)}\right)
\end{aligned}$$

[4]The case with the Hernquist potential is done Equation 4.128 of BT

[5]The number density $\nu(r)$ remains the same as before without the IMBH because it corresponds to the stellar mass density. The IMBH is not part of the stellar mass density that remains in colisionless equilibria. In fact, the IMBH is not allowed to move in this simple treatment, but it only provides with a source of radial acceleration (i.e., $d\phi/dr = GM_{\text{enc}}(r)/r^2$).

Problem 9: Toomre's criterion

a) In classes we saw that that axi-symmetric disturbances in a disk of the form $\exp(-i[\omega t + \mathbf{k} \cdot \mathbf{x}])$ **in follow a dispersion relation given by:**

$$\omega^2 = \kappa^2 - 2\pi G\Sigma|k| + v_s^2 k^2$$

Derive the Toomre Q dimensionless parameter to define a stable system. (Hint: find the most unstable wavenumber $|k|$ and impose stability in this worst-case scenario).

Given the dispersion relation we notice that if $\omega^2 > 0$, then ω is real and the disk is stable; while if $\omega^2 < 0$ then $\omega = \pm ip$ and the perturbation's amplitude would grow exponentially and the disk would be unstable. Then, stability in the worst-case scenario would be when $|k| = |k_{crit}|$ so that $\omega^2 = 0$:

$$\kappa^2 - 2\pi G\Sigma|k| + v_s^2 k^2 = 0 \quad \Rightarrow \quad |k_{crit}| = \frac{2\pi G\Sigma \pm \sqrt{(2\pi G\Sigma)^2 - 4v_s^2\kappa^2}}{2v_s^2}$$

Since k is the wave number, then it must be real, so:

$$(2\pi G\Sigma)^2 - 4v_s^2\kappa^2 > 0 \quad \Rightarrow \quad \frac{v_s^2\kappa^2}{\pi^2 G^2 \Sigma^2} > 1$$

So we can define the Toomre Q dimensionless parameter to define the stability of the system, which is stable if:

$$Q \equiv \frac{v_s \kappa}{\pi G \Sigma} > 1$$

b) Assuming that a disk has a sound speed $v_s = 1\ km/s\ (R/1au)^{-0.5}$ derive the critical density profile $\Sigma_c(R)$ such that the disk has $Q(R) = 1$. Disk with $\Sigma > \Sigma_c$ would be unstable to fragmentation. Assume that the central star has one Solar mass.

If the disk has $Q(R) = 1$, then

$$\Sigma_c = \frac{v_s \kappa}{\pi G}$$

In this case we can take

$$\kappa = \Omega = \sqrt{\frac{GM}{R^3}}$$

And so

$$\Sigma_c(R) = \frac{v_s}{\pi G}\sqrt{\frac{GM}{a^3}} = \frac{v_s}{\pi}\sqrt{\frac{M}{GR^3}}$$

If the central star has $M = M_\odot$, and the disk's sound speed is $v_s = 1 km/s\ (R/1au)^{-0.5}$ then

$$\Sigma_c(R) = \sqrt{\frac{M_\odot}{GR^3\pi^2}}\ km\ s^{-1}\sqrt{\frac{1au}{R}} = \sqrt{\frac{M_\odot}{GR^4\pi^2}}\ km\ s^{-1}(au)^{1/2}$$

c) [2 points] How much mass is enclosed between 1 au and 10 au?

To find the enclosed mass, we integrate the density profile:

$$M = \int \Sigma_c(R) R dR$$

Considering our previous result and a circular flat disk we have

$$M_{enclosed} = 2\pi \int_{1\ au}^{10\ au} \sqrt{\frac{M_\odot}{GR^4\pi^2}}\ km\ s^{-1}(au)^{1/2} R dR = 2\pi\sqrt{\frac{M_\odot}{G\pi^2}} \int_{1\ au}^{10\ au} R^{-1} dR\ km\ s^{-1}(au)^{1/2}$$

Then

$$M_{enclosed} = 2\sqrt{\frac{M_\odot}{G}} ln\left(\frac{10\ au}{1\ au}\right)\ km\ s^{-1}(au)^{1/2} \approx 0.155 M_\odot$$

Problem 10: Waves in weak bars

We shall complete the steps in section 3.3.3 to compute the motions of perturbations due to a weak bar. The treatment is reminiscent of the epicycle theory of nearly circular orbits.

a) From the Lagrangian in 3.134 in a rotating frame derive the equations of motion for R and ϕ (Eqs. 3.142a,b). Lay out the assumptions of the perturbative approach clearly.

From 3.134 we have

$$\mathcal{L} = \frac{1}{2}\dot{R}^2 + \frac{1}{2}[R(\dot{\varphi} + \Omega_b)]^2 - \Phi(R, \varphi)$$

We know from Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial R} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{R}}\right)$$

Where in this case

$$\frac{\partial \mathcal{L}}{\partial R} = R(\dot{\varphi} + \Omega_b)^2 - \frac{\partial \Phi}{\partial R}$$

and

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{R}}\right) = \ddot{R}$$

So

$$\boxed{\ddot{R} = R(\dot{\varphi} + \Omega_b)^2 - \frac{\partial \Phi}{\partial R}} \tag{9}$$

On the other hand,

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right)$$

Where

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -\frac{\partial \Phi}{\partial \varphi}$$

and

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) = \frac{d}{dt}\left(R^2(\dot{\varphi} + \Omega_b)\right)$$

So

$$\boxed{\frac{d}{dt}\left(R^2(\dot{\varphi} + \Omega_b)\right) = -\frac{\partial \Phi}{\partial \varphi}} \tag{10}$$

Considering a weak bar, we can apply perturbation theory so that we can write

$$\Phi(R, \varphi) = \Phi_0(R) + \Phi_1(R, \varphi)$$

So that

$$\frac{\partial \Phi}{\partial R} = \frac{d\Phi_0}{dR} + \frac{\partial \Phi_1}{\partial R}$$
$$\frac{\partial \Phi}{\partial \varphi} = \frac{\partial \Phi_1}{\partial \varphi}$$

We also can express our variables as

$$
\begin{aligned}
R(t) &= R_0 + R_1(t) \\
\varphi(t) &= \varphi_0(t) + \varphi_1(t)
\end{aligned}
$$

Then

$$
\begin{aligned}
\dot{R} &= \dot{R}_1 \\
\ddot{R} &= \ddot{R}_1 \\
\dot{\varphi} &= \dot{\varphi}_0 + \dot{\varphi}_1
\end{aligned}
$$

Replacing all this on Eq. 9 we have

$$
\begin{aligned}
\ddot{R}_1 &= (R_0 + R_1(t))(\dot{\varphi}_0 + \dot{\varphi}_1 + \Omega_b)^2 - \frac{d\Phi_0}{dR} - \frac{\partial \Phi_1}{\partial R} \\
&= R_0((\dot{\varphi}_0 + \Omega_b) + \dot{\varphi}_1)^2 + R_1(t)((\dot{\varphi}_0 + \Omega_b) + \dot{\varphi}_1)^2 - \frac{d\Phi_0}{dR} - \frac{\partial \Phi_1}{\partial R} \\
&= R_0(\dot{\varphi}_0 + \Omega_b)^2 + 2R_0\dot{\varphi}_1(\dot{\varphi}_0 + \Omega_b) + R_0\dot{\varphi}_1{}^2 + R_1(t)(\dot{\varphi}_0 + \dot{\varphi}_1 + \Omega_b)^2 - \frac{d\Phi_0}{dR} - \frac{\partial \Phi_1}{\partial R} \\
&= \left[R_0(\dot{\varphi}_0 + \Omega_b)^2 - \frac{d\Phi_0}{dR}\right] + \left[2R_0\dot{\varphi}_1(\dot{\varphi}_0 + \Omega_b) + R_0\dot{\varphi}_1{}^2 + R_1(t)(\dot{\varphi}_0 + \dot{\varphi}_1 + \Omega_b)^2 - \frac{\partial \Phi_1}{\partial R}\right]
\end{aligned}
$$

Where we separated zeroth-order terms from higher order terms. If we require that zeroth-order terms sum to zero we get

$$
R_0(\dot{\varphi}_0 + \Omega_b)^2 = \left(\frac{d\Phi_0}{dR}\right)_{R_0} \tag{11}
$$

And

$$
\ddot{R}_1 = 2R_0\dot{\varphi}_1(\dot{\varphi}_0 + \Omega_b) + R_0\dot{\varphi}_1{}^2 + R_1(t)(\dot{\varphi}_0 + \dot{\varphi}_1 + \Omega_b)^2 - \frac{\partial \Phi_1}{\partial R} \tag{12}
$$

On the other hand, for Eq. 10 we have

$$
\begin{aligned}
\frac{d}{dt}\big((R_0 + R_1(t))^2(\dot{\varphi}_0 + \dot{\varphi}_1 + \Omega_b)\big) &= -\frac{\partial \Phi_1}{\partial \varphi} \\
2\dot{R}_1(R_0 + R_1(t))(\dot{\varphi}_0 + \dot{\varphi}_1 + \Omega_b) + (R_0 + R_1(t))^2(\ddot{\varphi}_0 + \ddot{\varphi}_1 + \dot{\Omega}_b) &= -\frac{\partial \Phi_1}{\partial \varphi}
\end{aligned}
$$

Expanding the left-hand side of the expression, we get that the only zeroth-order term is $\ddot{\varphi}_0 R_0^2$, which means $\ddot{\varphi}_0 = 0 \Rightarrow \dot{\varphi}_0 = \text{constant}$ if we consider the same requirement as before. Or expression then becomes

$$
2\dot{R}_1(R_0 + R_1(t))(\dot{\varphi}_0 + \dot{\varphi}_1 + \Omega_b) + (R_0 + R_1(t))^2(\ddot{\varphi}_1 + \dot{\Omega}_b) = -\frac{\partial \Phi_1}{\partial \varphi} \tag{13}
$$

From Eq. 11 we can say

$$
\dot{\varphi}_0 + \Omega_b = \pm\sqrt{\frac{1}{R_0}\left(\frac{d\Phi_0}{dR}\right)_{R_0}}
$$

And we notice that

$$
\pm\sqrt{\frac{1}{R_0}\left(\frac{d\Phi_0}{dR}\right)_{R_0}} = \Omega(R_0) = \Omega_0
$$

So we have

$$\dot{\varphi}_0 = \Omega_0 - \Omega_b$$

We can choose the origin of time such that

$$\varphi_0(t) = (\Omega_0 - \Omega_b)t$$

Rewriting Eqs. 12 and 13 in terms of Ω_0 we have

$$\ddot{R}_1 = 2R_0\dot{\varphi}_1\Omega_0 + R_0\dot{\varphi}_1{}^2 + R_1(t)(\Omega_0 + \dot{\varphi}_1)^2 - \frac{\partial \Phi_1}{\partial R}$$

And

$$2\dot{R}_1(R_0 + R_1(t))(\Omega_0 + \dot{\varphi}_1) + (R_0 + R_1(t))^2(\ddot{\varphi}_1 + \dot{\Omega}_0) = -\frac{\partial \Phi_1}{\partial \varphi}$$

So finally, if we consider quadratic first-order terms as 0 and recall that $R = R_0 + R_1$, then we have

$$\boxed{\ddot{R}_1 + \left(\frac{d^2\Phi_0}{dR^2} - \Omega^2\right)_{R_0} R_1 - 2R_0\Omega_0\dot{\varphi}_1 = -\left(\frac{\partial \Phi_1}{\partial R}\right)_{R_0}}$$

$$\boxed{\ddot{\varphi}_1 + 2\Omega_0\frac{\dot{R}_1}{R_0} = -\frac{1}{R_0^2}\left(\frac{\partial \Phi_1}{\partial \varphi}\right)_{R_0}}$$

b) Assume that the bar potential is given by $\Phi_1 = \Phi_b(R)\cos(m\varphi)$ with $\varphi(t) = \varphi_0 + \varphi_1 = (\Omega - \Omega_b)t + \varphi_1$ and $\varphi_1 \ll 1$ and integrate the equations of motion to solve for $R_1(t)$. Where are the radial perturbations greatest?

We have

$$\Phi_1 = \Phi_b(R)\cos(m\varphi)$$

where

$$\varphi(t) = \varphi_1 + \varphi_2 = (\Omega_0 - \Omega_b)t + \varphi_1$$

If we assume $\varphi_1 \ll 1$, then we have

$$\varphi(t) \approx (\Omega_0 - \Omega_b)t$$

So that

$$\Phi_1 = \Phi_b(R)\cos(m(\Omega_0 - \Omega_b)t)$$

Then

$$\begin{aligned} \frac{\partial \Phi_1}{\partial R} &= \frac{d\Phi_b}{dR}\cos(m(\Omega_0 - \Omega_b)t) \\ \frac{\partial \Phi_1}{\partial \varphi} &= -m\Phi_b \sin(m(\Omega_0 - \Omega_b)t) \end{aligned}$$

Replacing these results in our equations of motion we get

$$\ddot{R}_1 + \left(\frac{d^2\Phi_0}{dR^2} - \Omega^2\right)_{R_0} R_1 - 2R_0\Omega_0\dot{\varphi}_1 = -\left(\frac{d\Phi_b}{dR}\right)_{R_0}\cos(m(\Omega_0 - \Omega_b)t) \tag{14}$$

$$\ddot{\varphi}_1 + 2\Omega_0 \frac{\dot{R}_1}{R_0} = \frac{m\Phi_b(R_0)}{R_0^2} \sin(m(\Omega_0 - \Omega_b)t) \tag{15}$$

Integrating Eq. 15 in time we have

$$\dot{\varphi}_1 = -2\Omega_0 \frac{R_1}{R_0} - \frac{\Phi_b(R_0)}{R_0^2(\Omega_0 - \Omega_b)} \cos(m(\Omega_0 - \Omega_b)t) + \mathcal{C}$$

Where $\mathcal{C}$ is a constant of integration. Replacing this into Eq. 14 we obtain

$$-\left(\frac{d\Phi_b}{dR}\right)_{R_0} \cos(m(\Omega_0 - \Omega_b)t) =$$
$$\ddot{R}_1 + \left(\frac{d^2\Phi_0}{dR^2} - \Omega^2\right)_{R_0} R_1 - 2R_0\Omega_0 \left(-2\Omega_0 \frac{R_1}{R_0} - \frac{\Phi_b(R_0)}{R_0^2(\Omega_0 - \Omega_b)} \cos(m(\Omega_0 - \Omega_b)t) + \mathcal{C}\right)$$

Reorganizing terms

$$\begin{aligned} \ddot{R}_1 + \left(\frac{d^2\Phi_0}{dR^2} - \Omega^2\right)_{R_0} R_1 + 4R_1\Omega_0^2 &= -\left[\frac{d\Phi_b}{dR} + \frac{2\Omega\Phi_b(R_0)}{R_0(\Omega_0 - \Omega_b)}\right]_{R_0} \cos(m(\Omega_0 - \Omega_b)t) + \mathcal{C} \\ \ddot{R}_1 + \kappa_0 R_1 &= -\left[\frac{d\Phi_b}{dR} + \frac{2\Omega\Phi_b}{R(\Omega - \Omega_b)}\right]_{R_0} \cos(m(\Omega_0 - \Omega_b)t) + \mathcal{C} \end{aligned} \tag{16}$$

Where

$$\kappa_0 = \left(\frac{d^2\Phi_0}{dR^2} + 3\Omega^2\right)_{R_0}$$

Eq. 9 is the equation of motion of a harmonic oscillator of natural frequency κ_0 that is driven at frequency $m(\Omega_0 - \Omega_b)$. The general solution to this equation is

$$R_1(t) = C_1 \cos(\kappa_0 t + \alpha) - \left[\frac{d\Phi_b}{dR} + \frac{2\Omega\Phi_b}{R(\Omega - \Omega_b)}\right]_{R_0} \frac{\cos[m(\Omega_0 - \Omega_b)t]}{\Delta} \tag{17}$$

Where C_1 and α are arbitrary constants and

$$\Delta \equiv \kappa_0^2 - m^2(\Omega_0 - \Omega_b)^2$$

Using that $\varphi_0(t) = (\Omega_0 - \Omega_b)t$ we find

$$R_1(\varphi_0) = C_1 \cos\left(\frac{\kappa_0\varphi_0}{\Omega_0 - \Omega_b} + \alpha\right) + C_2 \cos(m\varphi_0)$$

Where

$$C_2 = -\frac{1}{\Delta}\left[\frac{d\Phi_b}{dR} + \frac{2\Omega\Phi_b}{R(\Omega - \Omega_b)}\right]_{R_0}$$

The **perturbation is greatest** when $\Delta \to 0$. This occurs at the Lindblad resonances

$$\kappa_0^2 = m^2(\Omega_0 - \Omega_b)^2.$$

c) Recalling the motion from the epicycle theory, depict the motion for $m = 2$. Is the orbit closed? Ignore the free oscillations (i.e., set $C_1 = 0$).

If $C_1 = 0$ and $m = 2$ then $R_1(\varphi)$ becomes periodic in φ_0 with period $2\pi/2 = \pi$, so that

$$R_1(\varphi_0) = C_2 \cos(2(\Omega_0 - \Omega_b)t).$$

This correspond to two radial oscillations per one azimuthal period. Recalling that the radial coordinate is $R = R_0 + R_1(t)$, the resulting orbit is a large circle with an over imposed $m = 2$ lobe. For $R_1 \ll R_0$, the shape will be close to an ellipse centered at the origin. **Recall** from class that this is exactly what we wanted to beat the winding problem and produce nested ellipses that produce long-lived spirals.

Problem 11: Co-rotation and Lindblad resonances

We shall assume a simple model for the stellar disk in our galaxy following Miyamoto & Nagai:

$$\phi_{\mathrm{MN}}(\mathbf{R},\mathbf{z}) = -\frac{\mathbf{GM}}{\sqrt{\mathbf{R^2} + (\mathbf{a} + \sqrt{\mathbf{z^2}+\mathbf{b^2}})^2}} \tag{18}$$

with $M = 6.8 \times 10^{10} M_\odot$**,** $a = 3$ **kpc and** $b = 280$ **pc.**

a) Plot $\Omega(R)$**,** $\Omega(R) - \kappa(R)/2$ **and** $\Omega(R) + \kappa(R)/2$ **in the range** $1 - 20$ **kpc. Show your results in km/s/kpc. Which one decays more slowly with radius?**

From HW6 know that

$$\Omega(R) = \sqrt{\frac{GM}{(R^2 + (a+b)^2)^{3/2}}}$$

And

$$\kappa(R) = \sqrt{\frac{GM(4(a+b)^2 + R_g^2)}{(R_g^2 + (a+b)^2)^{5/2}}}$$

Then evaluating and plotting in Python we have

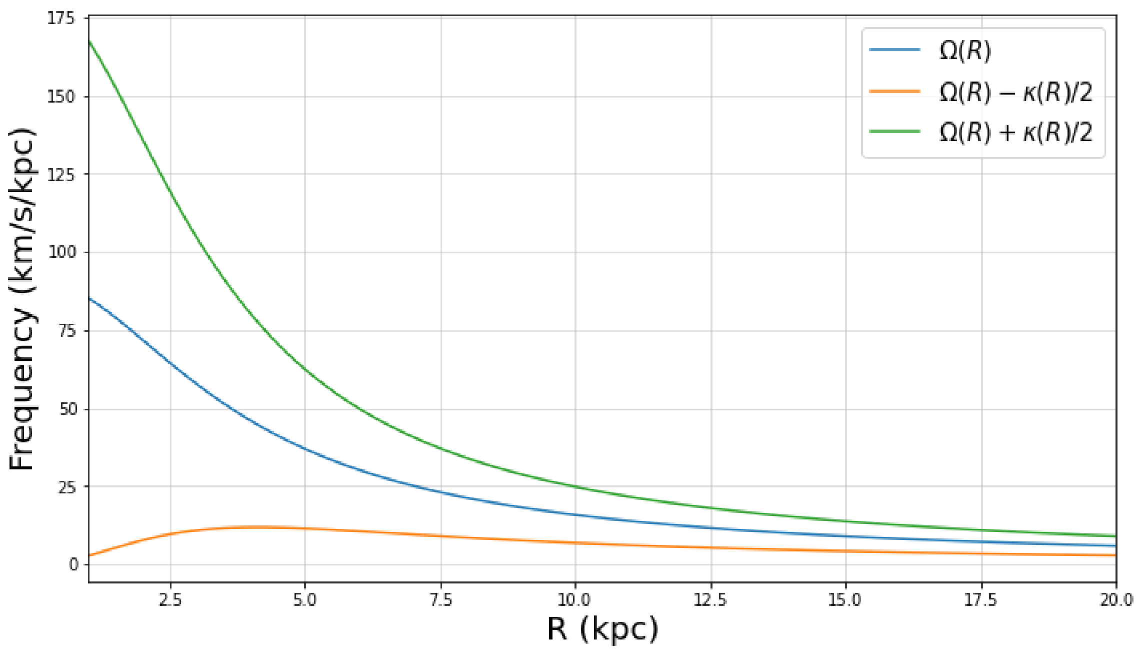


Figure 9: Frequencies as a function of R in the range 1-20 kpc

We can notice that $\Omega(R) - \kappa(R)/2$ initialy grows and then remains almost constant, decaying more slowly with radius than the other quantities.

b) Assuming that our Galaxy has a bar that rotates at a pattern speed $\Omega_p = 10$ **km/s/kpc, which resonances are present in the range** $1 - 20$ **kpc and where? Repeat for** $\Omega_p = 40$ **km/s/kpc.**

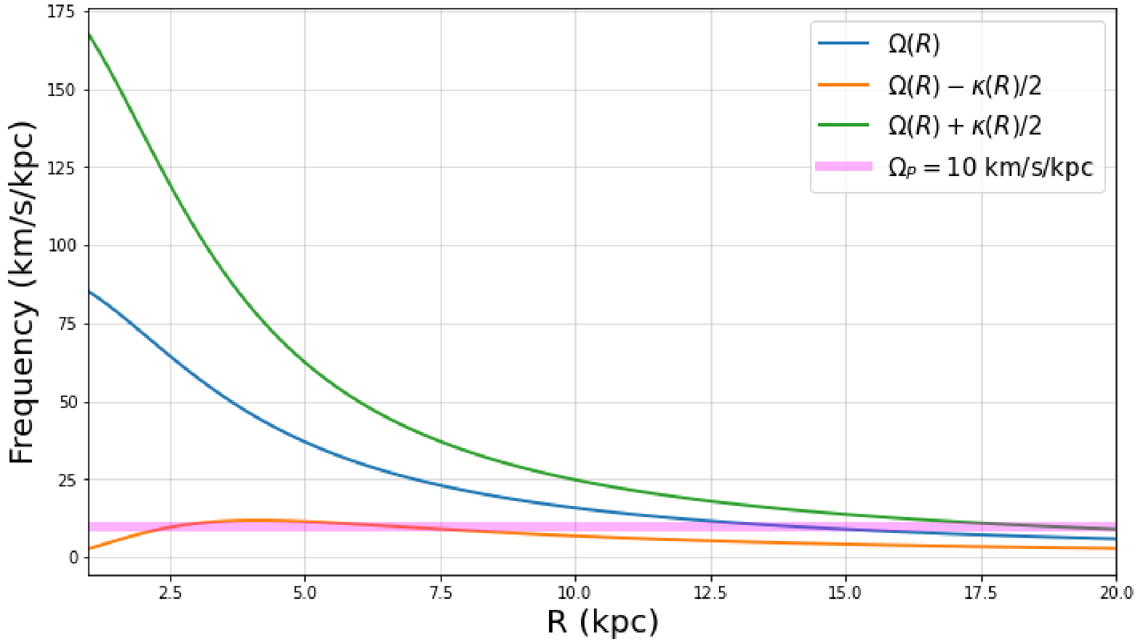


Figure 10: Frequencies as a function of R in the range 1-20 kpc along with a magenta line representing Ω_p

From Fig. 10 we notice that a corotation resonance can be found at $R \approx 14$ kpc, since $\Omega_P = \Omega(R)$ at that point. We also note that $\Omega_P = \Omega(R) - \kappa(R)/2$ at $R \approx 2.5$ kpc and $R \approx 6.8$ kpc, which indicates the presence of an inner $m = 2$ Lindblad resonance at those locations. On the other hand, at $R \approx 19$ kpc we find $\Omega_P = \Omega(R) + \kappa(R)/2$, corresponding to an outer $m = 2$ Lindblad resonance.

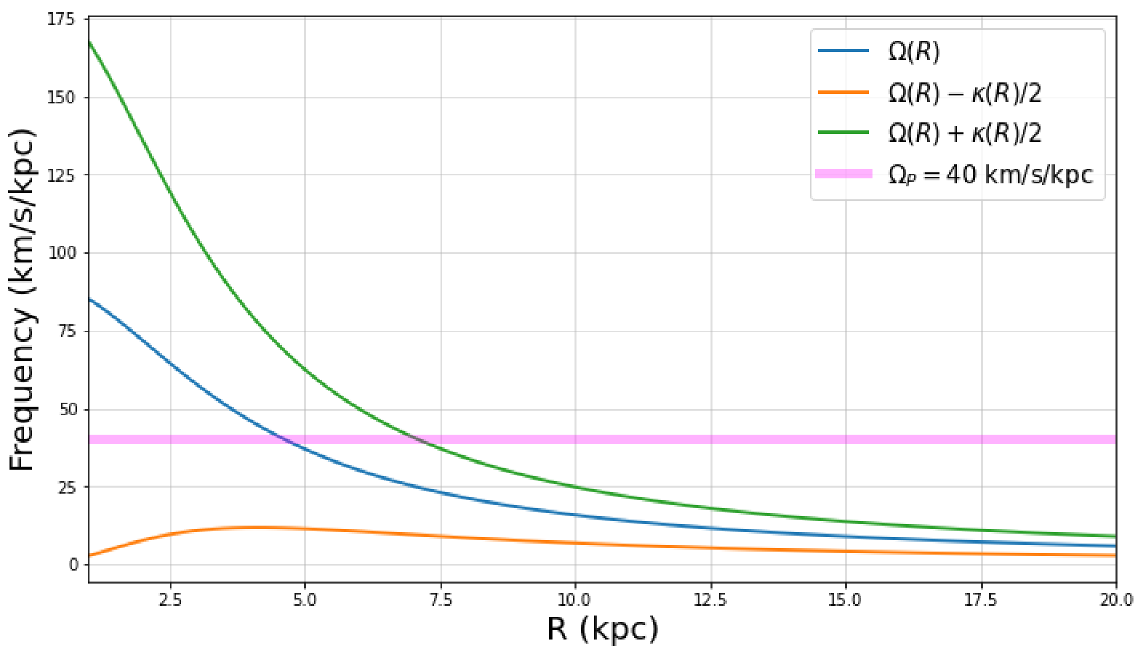


Figure 11: Frequencies as a function of R in the range 1-20 kpc along with a magenta line representing Ω_p

In this case we notice that $\Omega_P = \Omega(R)$ at $R \approx 4.8$ kpc, where we can find a corotation resonance. Also at $R \approx 7.3$ kpc we have $\Omega_P = \Omega(R) + \kappa(R)/2$, which means that an outer $m = 2$ Lindblad resonance is present.

Problem 12: Spherical Stellar System

We shall consider the motion of stars in the Galactic center. For simplicity we model the cluster as a uniform density sphere of mass M_c and radius a:

$$\rho_{\rm c}(r) = \begin{cases} \frac{3M_{\rm c}}{4\pi a^3} & r < a \\ 0 & r > a \end{cases}$$

At the center, there is a super-massive black hole of mass $M_\bullet$. Only consider the dynamics inside the cluster ($r < a$) in your arguments.

a. Compute the enclosed mass $M(r)$ and the circular velocity $V_c(r)$.

We know that for the cluster

$$M_{enc} = \int_0^r 4\pi r'^2 \rho(r')dr'$$

Then,

$$\begin{aligned} M_{enc} &= 4\pi \frac{3M_{\rm c}}{4\pi a^3} \int_0^r r'^2 dr' \\ &= \frac{3M_{\rm c}}{a^3}\frac{r^3}{3} \\ &= M_{\rm c}\left(\frac{r}{a}\right)^3 \end{aligned}$$

Where $r \le a$. Adding the mass of the black hole we have

$$M_{enc}(r) = M_{\rm c}\left(\frac{r}{a}\right)^3 + M_\bullet$$

And the circular velocity is given by

$$\begin{aligned} v_c(r) &= \sqrt{\frac{GM(r)}{r}} \\ &= \sqrt{\frac{G}{r}\left(M_{\rm c}\left(\frac{r}{a}\right)^3 + M_\bullet\right)} \\ &= \sqrt{\frac{GM_c r^2}{a^3} + \frac{GM_\bullet}{r}} \end{aligned}$$

b. Assuming colisionless equilibria, calculate the radial dispersion of velocities $\sigma_r^2(r)$. What is the velocity dispersion σ^2?

If we assume colisionless equilibria we can compute the radial dispersion of velocities using Jeans theorem:

$$\sigma_r^2 = \frac{1}{\nu(r)} \int_r^\infty \frac{GM_{enc}(r')\nu(r')}{r'^2} dr'$$

Given our density we have

$$\nu(r) = \frac{\rho(r)}{M_c} = \frac{3}{4\pi a^3}$$

Then

$$\sigma_r^2 = \frac{4\pi a^3}{3}\int_r^\infty \frac{G\left(M_\mathrm{c}\left(\frac{r}{a}\right)^3 + M_\bullet\right)}{r'^2}\frac{3}{4\pi a^3}dr'$$
$$= G\int_r^\infty \left(M_\mathrm{c}\frac{r'}{a^3} + \frac{M_\bullet}{r'^2}\right)dr'$$

Since we set $r \leq a$ we have

$$\sigma_r^2 = \frac{GM_\mathrm{c}}{a^3}\int_r^a r' + GM_\bullet\int_r^a \frac{1}{r'^2}dr'$$
$$= \frac{GM_\mathrm{c}}{a^3}\frac{r^2}{2}\Big|_r^a + GM_\bullet\frac{-1}{r}\Big|_r^a$$
$$= \frac{GM_\mathrm{c}}{a^3}\left(\frac{a^2}{2} - \frac{r^2}{2}\right) + GM_\bullet\left(\frac{1}{r} - \frac{1}{a}\right)$$
$$= G\left[\frac{M_c}{2a}\left(1 - \left(\frac{r}{a}\right)^2\right) + M_\bullet\left(\frac{1}{r} - \frac{1}{a}\right)\right]$$

And given the symmetry of our problem, $\sigma^2 = 3\sigma_r^2$

c. Calculate the azimuthal frequency $\Omega(r)$ and the epicyclic frequency $\kappa(r)$. What is the range of κ/Ω? Assume that $M_\bullet \ll M_c$

We know that

$$\Omega(r) = \frac{v_c}{r}$$

Then

$$\Omega(r) = \sqrt{\frac{GM_c}{a^3} + \frac{GM_\bullet}{r^3}}$$

From the epicyclic approximation we have

$$\kappa^2(r) = r\frac{\partial\Omega^2}{\partial r} + 4\Omega^2$$

So in this case

$$\kappa^2 = r\frac{\partial}{\partial r}\left(\frac{GM_c}{a^3} + \frac{GM_\bullet}{r^3}\right) + 4\left(\frac{GM_c}{a^3} + \frac{GM_\bullet}{r^3}\right)$$
$$= r\left(-3\frac{GM_\bullet}{r^4}\right) + 4\left(\frac{GM_c}{a^3} + \frac{GM_\bullet}{r^3}\right)$$
$$= -\frac{3GM_\bullet}{r^3} + 4\frac{GM_c}{a^3} + \frac{4HM_\bullet}{r^3}$$
$$= \frac{GM_\bullet}{r^3} + 4\frac{GM_c}{a^3}$$

So

$$\kappa = \sqrt{\frac{GM_\bullet}{r^3} + 4\frac{GM_c}{a^3}}$$

And

$$\frac{\kappa}{\Omega} = \sqrt{\frac{\frac{GM_\bullet}{r^3} + 4\frac{GM_c}{a^3}}{\frac{GM_c}{a^3} + \frac{GM_\bullet}{r^3}}}$$
$$= \sqrt{\frac{4M_c r^3 + M_\bullet a^3}{M_c r^3 + M_\bullet a^3}}$$

Near the center, $r \to 0$ so that the black hole dominates and

$$\frac{\kappa}{\Omega} = 1$$

If $M_\bullet \ll M_c$ then

$$\frac{\kappa}{\Omega} = 2$$

Finally

$$1 \leq \frac{\kappa}{\Omega} < 2$$

d. Find the locations where the orbits close. Depict an orbit at these locations.

The orbits close if

$$\frac{\kappa}{\Omega} = \frac{n}{m}$$

And then for the orbits to close

$$\frac{n^2}{m^2} = \frac{\frac{M_\bullet}{r^3} + 4\frac{M_c}{a^3}}{\frac{M_\bullet}{r^3} + \frac{M_c}{a^3}} = \frac{\mathcal{C}x^3 + 4}{\mathcal{C}x^3 + 1}$$

Where we defined

$$x = \frac{r}{a} \quad \text{and} \quad \mathcal{C} = \frac{M_\bullet}{M_c}$$

Then the locations are given by

$$n^2(\mathcal{C}x^3 + 1) = m^2(\mathcal{C}x^3 + 4)$$
$$\mathcal{C}x^3(n^2 - m^2) = 4m^2 - n^2$$
$$r = a\left(\frac{4m^2 - n^2}{\mathcal{C}(n^2 - m^2)}\right)^{1/3}$$

We could depict the orbits similar to Fig. 6.10 of BT:

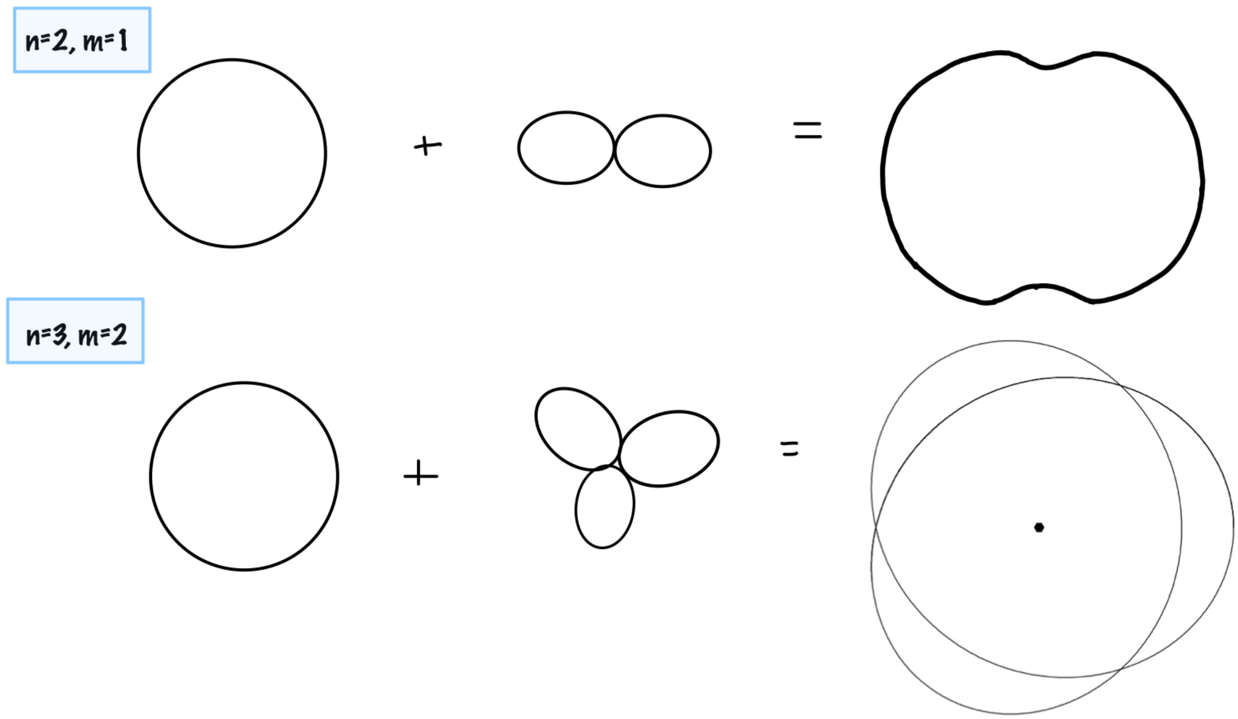


Figure 12: Closed orbits

Problem 13: Disk Dynamics

We shall model our Galaxy with a logarithmic potential

$$\phi(R) = \frac{V_0^2}{2} \ln(R_c^2 + R^2 + z^2/q^2). \tag{19}$$

a. Find the circular velocity at $z = 0$. What's it value at $R \gg R_c$ and how it compares its behavior with that of our Galaxy?

We know that

$$v_c(R) = \sqrt{R\frac{\partial \phi}{\partial R}}$$

For our logarithmic potential we have

$$\frac{\partial \phi}{\partial R} = \frac{RV_0^2}{R_c^2 + R^2 + \frac{z^2}{q^2}}$$

With $z = 0$,

$$v_c(R) = \sqrt{\frac{R^2 V_0^2}{R_c^2 + R^2}}$$

For $R \gg R_c$,

$$v_c(R) = \sqrt{\frac{R^2 V_0^2}{R^2}} = V_0$$

Which indicates a constant circular speed, consistent with the behaviour of our galaxy.

b. Compute $\Omega \pm \kappa/2$ and find the pattern speeds Ω_{inner} and Ω_{outer} such that the inner and outer Linblad resonances lie at $R = R_c$, respectively.

We know that

$$\Omega = \sqrt{\frac{1}{R}\frac{\partial \phi}{\partial R}}$$

Then

$$\Omega = \sqrt{\frac{V_0^2}{R_c^2 + R^2 + \frac{z^2}{q^2}}}$$

On the other hand,

$$\kappa^2 = R\frac{\partial \Omega^2}{\partial R} + 4\Omega^2$$

So in this case

$$\begin{aligned}
\kappa^2 &= R\frac{\partial}{\partial R}\left(\frac{V_0^2}{R_c^2 + R^2 + \frac{z^2}{q^2}}\right) + 4\left(\frac{V_0^2}{R_c^2 + R^2 + \frac{z^2}{q^2}}\right) \\
&= \frac{-2R^2V_0^2}{\left(R_c^2 + R^2 + \frac{z^2}{q^2}\right)^2} + 4\left(\frac{V_0^2}{R_c^2 + R^2 + \frac{z^2}{q^2}}\right) \\
&= \frac{V_0^2}{R_c^2 + R^2 + \frac{z^2}{q^2}}\left(4 - \frac{2R^2}{R_c^2 + R^2 + \frac{z^2}{q^2}}\right)
\end{aligned}$$

If we take $R = R_c$ and $z = 0$ then

$$\Omega \pm \frac{\kappa}{2} = \frac{V_0}{\sqrt{2}R_c} \pm \frac{V_0}{2R_c}\sqrt{\frac{3}{2}}$$

So that

$$\Omega_{inner} = \frac{V_0}{R_c\sqrt{2}}\left(1 - \frac{\sqrt{3}}{2}\right)$$
$$\Omega_{outer} = \frac{V_0}{R_c\sqrt{2}}\left(1 + \frac{\sqrt{3}}{2}\right)$$

c. Find q such that the stars around $R = R_c$ complete 2 cycles in the z direction per azimuthal oscillation. (Hint: compute $T_z = 2\pi/\nu$)

First we need to compute ν, and we know

$$\nu^2 = \frac{\partial^2\phi}{\partial z^2}$$

Then

$$\begin{aligned}
\nu^2 &= \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\frac{V_0^2}{2}ln\left(R_c^2 + R^2 + \frac{z^2}{q^2}\right)\right) \\
&= \frac{\partial}{\partial z}\left(\frac{V_0^2 \; 2z/q^2}{2\left(R_c^2 + R^2 + \frac{z^2}{q^2}\right)}\right) \\
&= \frac{V_0^2}{q^2}\frac{\partial}{\partial z}\left(\frac{z}{R_c^2 + R^2 + \frac{z^2}{q^2}}\right) \\
&= \frac{V_0^2}{q^2}\left(\frac{1}{R_c^2 + R^2 + \frac{z^2}{q^2}} - \frac{z \; 2z/q^2}{\left(R_c^2 + R^2 + \frac{z^2}{q^2}\right)^2}\right) \\
&= \frac{V_0^2}{q^2(R_c^2 + R^2) + z^2}\left(1 - \frac{2z^2}{q^2(R_c^2 + R^2) + z^2}\right)
\end{aligned}$$

Considering $R = R_c$ and $z = 0$ we have

$$\nu^2(R) = \frac{V_0}{\sqrt{2}qR}$$

From part b. we know

$$\Omega^2 = \frac{V_0^2}{R_c^2 + R^2 + \frac{z^2}{q^2}}$$

Then for $R = R_c$ and $z = 0$.

$$\Omega^2 = \frac{V_0}{\sqrt{2}R}$$

so that

$$\frac{\Omega}{\nu} = q$$

Considering that

$$T_\Omega = \frac{2\pi}{\Omega} \qquad T_z = \frac{2\pi}{\nu}$$

Then if the stars complete 2 cycles in the z direction per azimuthal oscillation,

$$\frac{T_z}{T_\Omega} = \frac{1}{2} \qquad \Rightarrow \frac{\Omega}{\nu} = \frac{1}{2}$$

And so $q = 1/2$ satisfies the condition.

d. We evaluate possible fragmentation of the disk at $R = R_c$. For simplicity we consider that $q = 1/\sqrt{2}$ so that Eq. 2.71c of BT for the density becomes

$$\rho(z) = \frac{3V_0^2}{8\pi G}\frac{R_c^2}{(R_c^2+z^2)^2} \quad \text{and} \quad \Sigma|_{R_c} = \int_{-\infty}^{\infty} dz\rho(z) = \frac{3V_0^2}{16GR_c} \tag{20}$$

Compute σ_z^2 at $z = 0$ using Jeans Equations (analogous to radial equilibria, Eq. 4.223). Then, using that $\sigma_R = \sigma_z$ compute the Toomre parameter of the disk.

From Eq. 4.223, considering z=0, we have

$$\bar{v_z^2} = \frac{1}{\rho(z=0)}\int_0^{\infty} dz'\rho(z')\frac{\partial\phi}{\partial z'}$$

We know that

$$\rho(z=0) = \frac{3V_0^2}{8\pi G R_c^2}$$

And at $R = R_c$ with $q = 1/\sqrt{2}$

$$\frac{\partial\phi}{\partial z} = \frac{2V_0^2 z}{2R_c^2 + 2z^2}$$

So

$$\begin{aligned}
\bar{v_z^2} &= \frac{8\pi G R_c^2}{3V_0^2}\int_0^{\infty}\frac{3V_0^2}{8\pi G}\frac{R_c^2}{(R_c^2+z'^2)^2}\cdot\frac{V_0^2 z'}{R_c^2+z'^2}dz' \\
&= R_c^4 V_0^2\int_0^{\infty}\frac{z'}{(R_c^2+z'^2)^3}dz' \\
&= R_c^4 V_0^2\left(-\frac{1}{4(R_c^2+z^2)^2}\bigg|_0^{\infty}\right) \\
&= \frac{V_0^2}{4}
\end{aligned}$$

And then $\sigma_z = \sigma_R = V_0/2$, so that using Eq. 6.71 (seen in class) and the previous result for κ we have:

$$Q = \frac{\sigma_R\kappa}{3.36G\Sigma} = \frac{\frac{V_0}{2}\sqrt{\frac{3}{2}}\frac{V_0}{R_c}}{3.36G}\frac{16GR_c}{3V_0^2} = \sqrt{\frac{3}{2}}\,\frac{8}{3.36\cdot 3} \approx 1$$

Problem 14: Relaxation

We shall estimate the probability that a star crosses a globular cluster of radius R_c. The star is "launched" at a distance R from the cluster with a velocity v pointing towards the center of the cluster. Its velocity is much greater than the random motions of the background stars, so you may assume that the relative velocities are simply v. Keep your solutions at the order-of-magnitude level (ignore 2, π, etc..)

a. Estimate the change in vertical velocity $\delta v_\perp$ when passing a star of mass m with closest approach b.

At closest approach

$$F_\perp \approx \frac{GMm}{b^2}$$

From Newton's second law,

$$F_\perp = m\frac{dv_\perp}{dt}$$

So that

$$\delta v_\perp = \frac{F_\perp}{m}\delta t$$

We know that the position of our star changes as

$$x = vt$$

so

$$\delta x = v\delta t$$

In this case $\delta x \approx b$ so $\delta t \approx \frac{b}{v}$ and then

$$\delta v_\perp \approx \frac{GM}{bv}$$

b. Estimate the rms orthogonal velocity $v_{\rm rms} = \sqrt{< v_\perp^2 >}$ once it reaches the cluster. Assume a constant number density n.

The number of encounters that our star suffers on its way to the cluster are given by

$$\delta N = \text{density number of stars} \times \text{dV}$$

We could consider that our star moves within a cylinder of height R and radius b, so that

$$\delta N = n \times 2\pi\ b\ db\ R$$

Then

$$\begin{aligned}
\left\langle v_\perp^2 \right\rangle &= \int_{-\infty}^{\infty} \delta v_\perp^2 \delta N \\
&= \int_{-\infty}^{\infty} \left(\frac{GM}{bv}\right)^2 n\ 2\pi\ b\ db\ R \\
&= \left(\frac{GM}{v}\right)^2 n\ 2\pi\ R\ \int_{-\infty}^{\infty} \frac{db}{b} \\
&= \left(\frac{GM}{v}\right)^2 n\ 2\pi\ R\ \ln\left(\frac{b_{max}}{b_{min}}\right)
\end{aligned}$$

Where $b_{max} = R$ and $b_{min} = b[90]$. Finally,

$$v_{rms} = \sqrt{\langle v_\perp^2 \rangle} = \sqrt{2\pi n R}\ \frac{GM}{v}\ \frac{1}{2} \ln \left(\frac{R}{b_{90}} \right)$$

Considering just the order of magnitude of our result we have

$$v_{rms} = \frac{GM}{v} (nR)^{1/2}$$

c. Provide with a condition on the initial velocity v so that star has a significant probability of crossing the cluster.

We know that, in total, the star will be deflected from its trajectory by

$$\Delta R_\perp = v_{rms}\ t_{cross}$$

where $t_{cross} = \frac{R}{v}$.
So that

$$\begin{aligned} \Delta R_\perp &= \frac{GM}{v} (nR)^{1/2} \frac{R}{v} \\ &= \frac{GM}{v^2} \sqrt{nR^3} \end{aligned}$$

If we want the star to cross the cluster, then we need $\Delta R_\perp \lesssim R_c$, so

$$\frac{GM}{v^2} \sqrt{nR^3} \lesssim R_c$$

And finally

$$v \gtrsim \sqrt{\frac{GM}{R_c}} (nR^3)^{1/4}$$